

DIMENSION FORMULAE IN GENUS ZERO AND UNIQUENESS OF VERTEX OPERATOR ALGEBRAS

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ABSTRACT. We prove a dimension formula for orbifold vertex operator algebras of central charge 24 by automorphisms of order n such that $\Gamma_0(n)$ is a genus zero group. We then use this formula together with the inverse orbifold construction for automorphisms of orders 2, 4, 5, 6 and 8 to establish that each of the following fifteen Lie algebras is the weight-one space V_1 of exactly one holomorphic, C_2 -cofinite vertex operator algebra V of CFT-type of central charge 24: $A_5C_5E_{6,2}$, $A_3A_{7,2}C_3^2$, $A_{8,2}F_{4,2}$, $B_8E_{8,2}$, $A_2^2A_{5,2}^2B_2$, $C_8F_4^2$, $A_{4,2}^2C_{4,2}$, $A_{2,2}^4D_{4,4}$, $B_5E_{7,2}F_4$, $B_4C_6^2$, $A_{4,5}^2$, $A_4A_{9,2}B_3$, B_6C_{10} , $A_1C_{5,3}G_{2,2}$ and $A_{1,2}A_{3,4}^3$.

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INTRODUCTION

Vertex operator algebras and their representation theory formalise the physical notions of chiral algebra and fusion ring of a conformal field theory (CFT). If the fusion ring is as simple as possible, then the conformal field theory is said to be holomorphic. More precisely, a rational vertex operator algebra is said to be *holomorphic* if it possesses a unique irreducible module, namely itself.

Throughout the paper, a vertex operator algebra is called *tame* if it is simple, rational, C_2 -cofinite, self-contragredient (or self-dual) and of CFT-type.

The central charge c of a tame, holomorphic vertex operator algebra V is necessarily a positive integer multiple of 8 (a simple consequence of [Zhu96]), and a complete classification is known in the cases $c = 8, 16$ [DM04a]. The classification at central charge $c = 24$ is an open problem, but a lot of progress has been made recently.

Let V be a tame, holomorphic vertex operator algebra and σ a finite order automorphism of V . In [EMS15] we study the fusion ring of the fixed-point vertex

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operator subalgebra V^σ , showing that it is naturally the group ring of a finite quadratic space and that Zhu's representation of $\mathrm{SL}_2(\mathbb{Z})$ on trace functions of irreducible V^σ -modules [Zhu96] coincides up to a character with the corresponding Weil representation. In the same paper we develop a cyclic orbifold theory and use it to construct a new holomorphic vertex operator algebra, denoted by $V^{\mathrm{orb}(\sigma)}$, for appropriately chosen σ .

To fully exploit the power of the orbifold theory as a tool for the construction and classification of holomorphic vertex operator algebras it is important to be able to deduce information about the isomorphism type of $V^{\mathrm{orb}(\sigma)}$ from V and σ . One useful invariant of a vertex operator algebra of central charge 24 is the dimension of its weight-one Lie algebra V_1 . In Theorem 3.1 below we prove a formula for the dimension of this Lie algebra $V_1^{\mathrm{orb}(\sigma)}$ in terms of V and σ . The following is a useful corollary. For more details see Section 3.

Theorem (Dimension Formula). *Let n be a positive integer such that the congruence subgroup $\Gamma_0(n) \subseteq \mathrm{SL}_2(\mathbb{Z})$ has genus zero. Let V be a tame, holomorphic vertex operator algebra of central charge 24 and σ an automorphism of V of type $n\{0\}$ such that V^σ satisfies the positivity assumption and such that the conformal weights of the σ^i -twisted V -modules obey $\rho(V(\sigma^i)) \geq 1$ for $i = 1, \dots, n-1$. Then*

$$\sum_{d|n} \frac{\phi((d, n/d))}{(d, n/d)} \left(24 + \frac{n}{d} \dim(V_1^\sigma) - \dim(V_1^{\mathrm{orb}(\sigma^d)}) \right) = 24$$

holds where ϕ is Euler's totient function.

In [EMS15] we also show that the weight-one space V_1 of a tame, holomorphic vertex operator algebra V of central charge $c = 24$ must be one of an explicit list of 71 Lie algebras, rigorously proving a result by Schellekens on “meromorphic $c = 24$ conformal field theories” [Sch93]. We shall in the following refer to these 71 cases as *Schellekens' list*.

Joint efforts by many authors have finally led to the result that all of these 71 Lie algebras on Schellekens' list are realised as weight-one spaces of tame, holomorphic vertex operator algebras of central charge 24 [FLM88, DGM90, Don93, DGM96, Lam11, LS12, LS15, Miy13, SS16, EMS15, Mö16, LS16b, LS16a, LL16]. All are obtained as lattice vertex operator algebras and their (iterated) orbifolds by finite order automorphisms. Some of these results are based on the general theory of cyclic orbifolds developed in [EMS15, Mö16].

In order to obtain a classification of vertex operator algebras it is necessary to show that the Lie algebra structure of the weight-one space of a tame, holomorphic vertex operator algebra of central charge 24 uniquely determines this vertex operator algebra up to isomorphism. This is known for the 24 Niemeier lattice vertex operator algebras [DM04a], 3 vertex operator algebras obtained as orbifolds of order 3 [LS16c], 13 vertex operator algebras obtained as (-1) -orbifolds [KLL16], 2 framed vertex operator algebras [LS15] and for one more case [LL16].

In this paper we add 14 more cases to this list (see the following theorem), which leaves 14 open cases, including the Moonshine module V^\natural . It is a decades-old conjecture that V^\natural is the only (tame) holomorphic vertex operator algebra V of central charge 24 with $V_1 = \{0\}$ [FLM88]. The methods used in this paper are, however, unsuited to prove this conjecture.

Theorem (Uniqueness). *A tame, holomorphic vertex operator algebra of central charge 24 is up to isomorphism uniquely determined by the Lie algebra structure of its weight-one space V_1 if V_1 is isomorphic to $A_5C_5E_{6,2}$, $A_3A_{7,2}C_3^2$, $A_{8,2}F_{4,2}$, $B_8E_{8,2}$, $A_2^2A_{5,2}B_2$, $C_8F_4^2$, $A_{4,2}^2C_{4,2}$, $A_{2,2}^4D_{4,4}$, $B_5E_{7,2}F_4$, $B_4C_6^2$, $A_{4,5}^2$, $A_4A_{9,2}B_3$,*

B_6C_{10} , $A_1C_{5,3}G_{2,2}$ or $A_{1,2}A_{3,4}^3$, corresponding to the cases 44, 33, 36, 62, 26, 52, 22, 13, 53, 48, 9, 40, 56, 21 and 7 on Schellekens' list.

The strategy we use to prove uniqueness is laid out in [LS16c] and is based in an essential way on the *inverse orbifold construction*, first developed in [EMS15] (and called “reverse orbifold” in [LS16c]). In a nutshell, the idea is as follows: Let V be a tame, holomorphic vertex operator algebra of central charge 24 with weight-one Lie algebra V_1 . Any inner automorphism $\sigma = e^{(2\pi i) \text{ad}_v}$ of V_1 extends to an automorphism of V . Choose such an automorphism, and suppose that $U = V^{\text{orb}(\sigma)}$ is isomorphic to the lattice vertex operator algebra V_L . By virtue of its construction as an orbifold, U carries an inverse orbifold automorphism ζ of the same order as σ with $V \cong U^{\text{orb}(\zeta)}$. Under favourable circumstances the order and the fixed-point subalgebra $U_1^\zeta = V_1^\sigma$ of ζ determine it up to conjugacy in $\text{Aut}(V_L)$. If this is the case, then $V \cong (V_L)^{\text{orb}(\zeta)}$ is shown to be unique up to isomorphism.

Outline. We assume that the reader is familiar with vertex operator algebras and their representation theory and with the specific examples provided by lattice vertex operator algebras and affine vertex operator algebras (see e.g. [LL04]).

The article is organised as follows: In Section 1 we describe Lie algebras occurring as weight-one spaces of vertex operator algebras, inner automorphisms and the classification of the weight-one structures of tame, holomorphic vertex operator algebras of central charge 24, dubbed Schellekens' list.

Section 2 reviews the cyclic orbifold construction of tame, holomorphic vertex operator algebras and the inverse orbifold construction. In Section 3 a genus zero dimension formula for the weight-one space of these orbifolds for central charge 24 is derived.

In Section 4 the general procedure to prove the uniqueness of certain vertex operator algebras on Schellekens' list is described. This is split up into three parts which are described in detail in Sections 5, 6 and 7 for the fifteen cases considered in this text.

Finally, in Section 8 we state the main result of this paper.

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1. VERTEX OPERATOR ALGEBRAS AND LIE ALGEBRAS

We briefly review the appearance of Lie algebras as weight-one spaces of vertex operator algebras and discuss related concepts such as affine structures and inner automorphisms.

1.1. Lie Algebras. Let \mathfrak{g} be a finite-dimensional, simple Lie algebra over \mathbb{C} of rank $l \in \mathbb{Z}_{>0}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, $\Phi \subseteq \mathfrak{h}^*$ a root system and $\Delta = \{\alpha_1, \dots, \alpha_l\} \subseteq \Phi$ a base of simple roots, labelled as in [Hum73], p. 58. We denote by (\cdot, \cdot) the unique non-degenerate, symmetric, invariant bilinear form on \mathfrak{g} normalised so that the induced form on \mathfrak{h}^* , which we also denote by (\cdot, \cdot) , satisfies $(\alpha, \alpha) = 2$ for any long root $\alpha \in \Phi$.

The coroot $\alpha^\vee \in \mathfrak{h}$ associated with $\alpha \in \Phi$ is defined by $\beta(\alpha^\vee) = 2(\alpha, \beta)/(\alpha, \alpha)$ for all $\beta \in \mathfrak{h}^*$. The set $\Phi^\vee \subseteq \mathfrak{h}$ of coroots is also a root system, of which the set $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ is a base. The root lattice Q and the coroot lattice Q^\vee are the \mathbb{Z} -spans of Φ and Φ^\vee , respectively.

The lattice $P \subseteq \mathfrak{h}^*$ canonically dual to $Q^\vee \subseteq \mathfrak{h}$ is called the weight lattice, and the elements of the basis $\Lambda_1, \dots, \Lambda_l \in \mathfrak{h}^*$ dual to the simple coroots $\alpha_1^\vee, \dots, \alpha_l^\vee \in \mathfrak{h}$ are called the fundamental weights. Similarly, the fundamental coweights $\Lambda_1^\vee, \dots, \Lambda_l^\vee \in \mathfrak{h}$ are defined to be dual to the simple roots and span the coweight lattice. By definition $2(\Lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \Lambda_i(\alpha_j^\vee) = \delta_{i,j}$ holds for $i, j = 1, \dots, l$.

Let $\delta := \sum_{i=1}^l \Lambda_i$ be the sum of all fundamental weights and $\delta^\vee := \sum_{i=1}^l \Lambda_i^\vee$ the sum of the fundamental coweights. Among the roots $\alpha = \sum_{i=1}^l m_i \alpha_i$ of \mathfrak{g} , the unique one with the highest value of $\sum_{i=1}^l m_i$ is denoted θ and called highest root. The highest root is always a long root. The relation $\theta = \sum_{i=1}^l a_i \alpha_i$ defines the “marks” a_1, \dots, a_l of \mathfrak{g} . Similarly, the corresponding coroot θ^\vee defines the “comarks” $a_1^\vee, \dots, a_l^\vee$ of \mathfrak{g} by $\theta^\vee = \sum_{i=1}^l a_i^\vee \alpha_i^\vee$. The Coxeter number and dual Coxeter number are defined as $h := 1 + \sum_{i=1}^l a_i$ and $h^\vee := 1 + \sum_{i=1}^l a_i^\vee$.

We denote by $P_+ \subseteq \mathfrak{h}^*$ the set of dominant integral weights, i.e. the non-negative integer linear combinations of the fundamental weights Λ_i . The level of a weight $\mu \in P_+$ is by definition $\mu(\theta^\vee)$. For $k \in \mathbb{Z}_{\geq 0}$ let $P_+^k := \{\mu \in P_+ \mid \mu(\theta^\vee) \leq k\}$ denote the subset of P_+ of dominant integral weights of level at most k . Note that the level of δ is $h^\vee - 1$.

The reflection r_α in the root $\alpha \in \Phi$ is the isometry of \mathfrak{h}^* given by $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ for all $\beta \in \mathfrak{h}^*$, and the Weyl group $W \subseteq \text{GL}(\mathfrak{h}^*)$ is the group generated by the reflections in all roots or equivalently in all simple roots. W fixes Φ , Q and P as sets. Via the isometry $\mathfrak{h} \rightarrow \mathfrak{h}^*$ the Weyl group W also acts on \mathfrak{h} and fixes Φ^\vee , Q^\vee and the coweight lattice.

We will sometimes write $\Phi(\mathfrak{g})$, $\Delta(\mathfrak{g})$, $(\cdot, \cdot)_{\mathfrak{g}}$, etc. in order to specify the simple Lie algebra \mathfrak{g} .

For a dominant integral weight $\lambda \in P_+$ let $L_{\mathfrak{g}}(\lambda)$ be the up to isomorphism unique finite-dimensional, irreducible highest-weight \mathfrak{g} -module with highest weight λ . For any finite-dimensional \mathfrak{g} -module M we denote by $\Pi(M) \subseteq P$ the set of weights of M , i.e. M has weight decomposition $M = \bigoplus_{\mu \in \Pi(M)} M_\mu$. We also denote $\Pi(L_{\mathfrak{g}}(\lambda))$ by $\Pi(\lambda, \mathfrak{g})$.

1.2. Weight-One Lie Algebra. Let $V = \bigoplus_{n=0}^\infty V_n$ be a vertex operator algebra of CFT-type. It is well-known that

$$[a, b] := a_0 b$$

for $a, b \in V_1$ equips V_1 with the structure of a Lie algebra and that the zero modes a_0 of $a \in V_1$ equip each V -module with a weight-preserving action of this Lie algebra.

If V is also self-contragredient, then there exists a non-degenerate, invariant bilinear form $\langle \cdot, \cdot \rangle$ on V , which is unique up to a non-zero scalar and symmetric [Li94]. We normalise the form so that $\langle \mathbf{1}, \mathbf{1} \rangle = -1$, where $\mathbf{1}$ is the vacuum vector of V . Then, for $a, b \in V_1$,

$$a_1 b = b_1 a = \langle a, b \rangle \mathbf{1}.$$

Note that V_n and V_m are orthogonal with respect to $\langle \cdot, \cdot \rangle$ for $m \neq n$.

If $g \in \text{Aut}(V)$ is an automorphism of the vertex operator algebra V (which by definition fixes the vacuum vector $\mathbf{1} \in V_0$ and the Virasoro vector $\omega \in V_2$), then the restriction of g to V_1 is a Lie algebra automorphism, possibly of smaller order.

1.3. Affine Structure. Let \mathfrak{g} be a simple, finite-dimensional Lie algebra with invariant bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$, normalised as described above. Then the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated with \mathfrak{g} is the Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with central element K and Lie bracket

$$(1) \quad [a \otimes t^m, b \otimes t^n] := [a, b] \otimes t^{m+n} + m(a, b)_{\mathfrak{g}} \delta_{m+n, 0} K$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

A representation of $\hat{\mathfrak{g}}$ is said to have level $k \in \mathbb{C}$ if K acts as $k \text{ id}$. For a dominant integral weight $\lambda \in P_+(\mathfrak{g})$ and $k \in \mathbb{C}$ let $L_{\hat{\mathfrak{g}}}(k, \lambda)$ be the irreducible $\hat{\mathfrak{g}}$ -module of level k obtained by inducing the irreducible highest-weight \mathfrak{g} -module $L_{\mathfrak{g}}(\lambda)$ up in a certain way to a $\hat{\mathfrak{g}}$ -module and taking its irreducible quotient. For more details on affine Kac-Moody algebras and their modules we refer the reader to [Kac90].

For $k \in \mathbb{Z}_{>0}$, $L_{\hat{\mathfrak{g}}}(k, 0)$ admits the structure of a rational vertex operator algebra whose irreducible modules are given by the $L_{\hat{\mathfrak{g}}}(k, \lambda)$ for $\lambda \in P_+^k(\mathfrak{g})$, the subset of the dominant integral weights $P_+(\mathfrak{g})$ of level at most k (see [FZ92], Theorem 3.1.3 or e.g. [LL04], Sections 6.2 and 6.6). The conformal weight of $L_{\hat{\mathfrak{g}}}(k, \lambda)$ is

$$\rho(L_{\hat{\mathfrak{g}}}(k, \lambda)) = \frac{(\lambda + 2\delta, \lambda)_{\mathfrak{g}}}{2(k + h^{\vee})}$$

(see [Kac90], Corollary 12.8).

For a self-contragredient vertex operator algebra V of CFT-type

$$[a_m, b_n] = (a_0 b)_{m+n} + m(a_1 b)_{m+n-1} = [a, b]_{m+n} + m\langle a, b \rangle \delta_{m+n,0} \text{id}_V$$

for all $a, b \in V_1$, $m, n \in \mathbb{Z}$. Comparing this with (1) we see that for a simple Lie subalgebra \mathfrak{g} of V_1 the map $a \otimes t^n \mapsto a_n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ defines a representation of $\hat{\mathfrak{g}}$ on V of some level $k_{\mathfrak{g}} \in \mathbb{C}$ with $\langle \cdot, \cdot \rangle = k_{\mathfrak{g}}(\cdot, \cdot)_{\mathfrak{g}}$ restricted to \mathfrak{g} .

In the following let us assume that V is tame. Then it is shown in [DM04b] that the Lie algebra V_1 is reductive, i.e. a direct sum of a semisimple and an abelian Lie algebra. Moreover, Theorem 3.1 in [DM06] states that for a simple Lie subalgebra \mathfrak{g} of V_1 the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{g} is non-degenerate, the level $k_{\mathfrak{g}}$ is a positive integer, the vertex operator subalgebra of V generated by \mathfrak{g} is isomorphic to $L_{\hat{\mathfrak{g}}}(k_{\mathfrak{g}}, 0)$ and V is an integrable $\hat{\mathfrak{g}}$ -module.

Now suppose that $c = 24$. It is shown in [DM04a] that the Lie algebra V_1 is zero, abelian of dimension 24 or semisimple of rank at most 24. Let us assume that V_1 is semisimple. Then V_1 decomposes into a direct sum

$$V_1 = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$$

of simple Lie algebras \mathfrak{g}_i and the vertex operator subalgebra $\langle V_1 \rangle$ of V generated by V_1 is isomorphic to

$$\langle V_1 \rangle \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, 0)$$

with levels $k_i := k_{\mathfrak{g}_i} \in \mathbb{Z}_{>0}$ and has the same Virasoro vector as V . This decomposition of the vertex operator algebra $\langle V_1 \rangle$ is called the *affine structure* of V . We denote it by the symbol

$$\mathfrak{g}_{1,k_1} \dots \mathfrak{g}_{r,k_r}$$

and sometimes omit k_i if it equals 1. By abuse of language we will also refer to the cases of abelian and zero Lie algebras V_1 as certain affine structures.

Since $\langle V_1 \rangle \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, 0)$ is rational, V decomposes into the direct sum of finitely many irreducible $\langle V_1 \rangle$ -modules

$$(2) \quad V \cong \bigoplus_{(\lambda_1, \dots, \lambda_r)} m_{(\lambda_1, \dots, \lambda_r)} L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, \lambda_r)$$

where the $m_{(\lambda_1, \dots, \lambda_r)} \in \mathbb{Z}_{\geq 0}$ and the sum runs over finitely many tuples $(\lambda_1, \dots, \lambda_r)$ of dominant integral weights with each $\lambda_i \in P_+^{k_i}(\mathfrak{g}_i)$, i.e. of level at most k_i . Note that the conformal weight of a module $M = L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, \lambda_r)$ is

$$(3) \quad \rho(M) = \sum_{i=1}^r \frac{(\lambda_i + 2\delta_i, \lambda_i)_{\mathfrak{g}_i}}{2(k_i + h_i^{\vee})}.$$

Since $\langle V_1 \rangle_1 = V_1$, except for $(\lambda_1, \dots, \lambda_r) = (0, \dots, 0)$ the conformal weights of the tensor-product modules in the above decomposition have to be at least 2.

1.4. Schellekens' List. In [Sch93] Schellekens classified the possible affine structures of “meromorphic conformal field theories of central charge 24”. In [EMS15] his result is proved as a rigorous theorem on vertex operator algebras:

Theorem 1.1 ([EMS15], Theorem 6.4, [Sch93]). *Let V be a tame, holomorphic vertex operator algebra of central charge 24. Then one of the following holds:*

- (1) $V_1 = 0$,
- (2) V_1 is the 24-dimensional abelian Lie algebra \mathbb{C}^{24} (and V is isomorphic to the vertex operator algebra associated with the Leech lattice) or
- (3) V_1 is one of 69 semisimple Lie algebras with a certain affine structure given in Table 1 of [Sch93].

We remark that entry 62 in Table 1 of [Sch93] should read $E_{8,2}B_{8,1}$ (this typo is corrected in the arXiv-version of the paper).

Idea of Proof. The modular invariance of the character of V imposes strong constraints on the Lie algebra V_1 and its module V_2 . In particular, V_1 is either trivial, abelian of dimension 24 or semisimple [DM04a]. Suppose in the following that V_1 is semisimple.

Let \mathfrak{g} be a simple Lie algebra and M a finite-dimensional \mathfrak{g} -module with weight decomposition $M = \bigoplus_{\mu \in \Pi(M)} M_\mu$. We consider the power sums

$$S_M^j(z) := \sum_{\mu \in \Pi(M)} \dim(M_\mu) \mu(z)^j$$

for $j \in \mathbb{Z}_{\geq 0}$, which are polynomials in $z \in \mathfrak{h}$. This definition generalises straightforwardly to semisimple \mathfrak{g} .

Modular invariance of the character of V implies the power sum identity

$$S_{V_1}^2(z) = \frac{\dim(V_1) - 24}{12} \langle z, z \rangle,$$

which in turn implies [DM04a]

$$(4) \quad \frac{h_i^\vee}{k_i} = \frac{\dim(V_1) - 24}{24}$$

for each simple component \mathfrak{g}_i of V_1 where h_i^\vee is the dual Coxeter number of \mathfrak{g}_i . Equation (4) has 221 solutions (see [EMS15], Proposition 6.3, [Sch93]).

Modular invariance implies further power sum identities on V_2 , for example

$$S_{V_2}^2(z) = (32808 - 2 \dim(V_1)) \langle z, z \rangle.$$

In [EMS15], Theorem 6.1, a list of these identities is presented. They are strong enough to constrain V_1 to be one of an explicit list of 69 semisimple Lie algebras. Together with the trivial and abelian case for V_1 these give the 71 affine structures in Schellekens' list (see [Sch93] and [EMS15], Theorem 6.4). \square

Joint efforts to construct all the cases on Schellekens' list [FLM88, DGM90, Don93, DGM96, Lam11, LS12, LS15, Miy13, SS16, EMS15, Mö16, LS16b, LS16a, LL16] have finally led to:

Theorem 1.2. *A tame, holomorphic vertex operator algebra of central charge 24 possesses one of exactly 71 affine structures, namely the 71 cases on Schellekens' list.*

An actual classification of vertex operator algebras can be obtained if the following uniqueness conjecture is proved:

Conjecture 1.3. *Let V be a tame, holomorphic vertex operator algebra of central charge 24. Then the Lie algebra structure of V_1 uniquely determines the vertex operator algebra V up to isomorphism.*

In this text we will prove the uniqueness conjecture for fifteen cases (fourteen of them new¹) on Schellekens' list.

Theorem (Uniqueness). *A tame, holomorphic vertex operator algebra of central charge 24 is up to isomorphism uniquely determined by the Lie algebra structure of its weight-one space V_1 if V_1 is isomorphic to $A_5C_5E_{6,2}$, $A_3A_{7,2}C_3^2$, $A_{8,2}F_{4,2}$, $B_8E_{8,2}$, $A_2^2A_{5,2}B_2$, $C_8F_4^2$, $A_{4,2}^2C_{4,2}$, $A_{2,2}^4D_{4,4}$, $B_5E_{7,2}F_4$, $B_4C_6^2$, $A_{4,5}^2$, $A_4A_{9,2}B_3$, B_6C_{10} , $A_1C_{5,3}G_{2,2}$ or $A_{1,2}A_{3,4}^3$, corresponding to the cases 44, 33, 36, 62, 26, 52, 22, 13, 53, 48, 9, 40, 56, 21 and 7 on Schellekens' list.*

1.5. Lattice Vertex Operator Algebras. To keep the exposition self-contained we recall a few well-known facts about lattice vertex operator algebras [Bor86, FLM88, Don93].

Let L be a positive-definite, even lattice, i.e. a free abelian group L of finite rank r equipped with a positive-definite, symmetric bilinear form $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z}$ such that the norm $\langle \alpha, \alpha \rangle / 2$ is an integer for all $\alpha \in L$. The lattice vertex operator algebra V_L constructed from L is tame, has central charge $c = r$ and carries a grading $V_L = \bigoplus_{\alpha \in L} V_L(\alpha)$ by L . If L is unimodular, then V_L is holomorphic.

The complexified lattice $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$ can be naturally identified with $\mathfrak{h} := \{h(-1) \otimes \mathfrak{e}_0 \mid h \in L_{\mathbb{C}}\} \subseteq (V_L)_1$, which is a Cartan subalgebra for the reductive Lie algebra $(V_L)_1$.

The construction of V_L involves a choice of group 2-cocycle $\varepsilon : L \times L \rightarrow \{\pm 1\}$. An automorphism $\nu \in \text{Aut}(L)$ together with a 1-cocycle $\eta : L \rightarrow \{\pm 1\}$ satisfying

$$\eta(\alpha)\eta(\beta)/\eta(\alpha + \beta) = \varepsilon(\alpha, \beta)/\varepsilon(\nu\alpha, \nu\beta)$$

defines an automorphism $\hat{\nu} \in \text{Aut}(V_L)$. We call $\hat{\nu}$ a *standard lift* if the restriction of η to the fixed-point sublattice $L^{\nu} \subseteq L$ is trivial. Let the order of ν be m . If m is odd or if m is even and $\langle \alpha, \nu^{m/2}\alpha \rangle$ is even for all $\alpha \in L$, then the order of $\hat{\nu}$ is also m . Otherwise the order of $\hat{\nu}$ is $2m$, in which case we say ν exhibits order doubling.

There are precisely 24 positive-definite, even, unimodular lattices of rank 24, the Niemeier lattices. Each is determined up to isomorphism by its root lattice, and we write $N(Q)$ for the Niemeier lattice with non-empty root lattice Q . The one remaining Niemeier lattice, the Leech lattice, is denoted by Λ .

1.6. Inner Automorphisms. We describe inner automorphisms of vertex operator algebras and the twisted modules associated with them. This summary is based on [Li96], Section 5. Let V be a vertex operator algebra of CFT-type. We define $K := \langle \{e^{v_0} \mid v \in V_1\} \rangle \subseteq \text{Aut}(V)$ to be the *inner automorphism group* of V . Note that since V is of CFT-type, $v_0\omega = 0$ for all $v \in V_1$ so that the inner automorphisms preserve the Virasoro vector ω , which we include in our definition of vertex operator algebra automorphism.

Let $v \in V_1$ such that the zeroth mode v_0 acts semisimply on V . Then the inner automorphism $\sigma_v := e^{-(2\pi i)v_0}$ fulfils $\sigma_v^T = 1$ for some $T \in \mathbb{Z}_{>0}$ if and only if all eigenvalues of v_0 lie in $(1/T)\mathbb{Z}$.

Note that by definition of the Lie algebra structure on V_1 , σ_v restricts to the inner automorphism $e^{-(2\pi i)\text{ad}_v} \in \text{Inn}(V_1) \subseteq \text{Aut}(V_1)$ of V_1 .

Now, let $\sigma \in \text{Aut}(V)$ be an automorphism of V such that $\sigma v = v$. Then σ and σ_v commute. We also assume that $L_1 v = 0$. If V is self-contragredient and of CFT-type, then by results in [Li94], $L_1 V_1 = \{0\}$ and the assumption holds for any $v \in V_1$.

¹The uniqueness of case $B_8E_{8,2}$ (case 62) is already proved in [LS15] using framed vertex operator algebras.

Suppose that $M = (M, Y_M)$ is a σ -twisted V -module. We consider Li's operator

$$\Delta_v(z) := z^{v_0} \exp \left(\sum_{n=1}^{\infty} \frac{v_n}{-n} (-z)^{-n} \right)$$

and define $M^{(v)} = (M^{(v)}, Y_{M^{(v)}})$ as $M^{(v)} := M$ (as vector space) and

$$Y_{M^{(v)}}(a, z) := Y_M(\Delta_v(z)a, z)$$

for all $a \in V$. Then $M^{(v)}$ is a $\sigma_v \sigma$ -twisted V -module, which is irreducible if and only if M is.²

The L_0 -grading of $M^{(v)}$ is shifted in comparison to M via

$$L_0^{M^{(v)}} = L_0^M + v_0 + \frac{1}{2}(v_1 v)_{-1}$$

where the v_n , $n \in \mathbb{Z}$, denote the modes of the unmodified module vertex operator $Y_M(\cdot, z)$. If V is self-contragredient, then we can rewrite the last term and obtain

$$(5) \quad L_0^{M^{(v)}} = L_0^M + v_0 + \frac{1}{2} \langle v, v \rangle \text{id}_M$$

using the symmetric, invariant bilinear form $\langle \cdot, \cdot \rangle$ on V .

1.7. Kac's Theory. We briefly describe a beautiful theory by Kac on the conjugacy classes of automorphisms of simple Lie algebras and the associated fixed-point Lie subalgebras (see Chapter 8 in [Kac90]). In Theorem 8.6 Kac classifies the finite-order automorphisms of simple Lie algebras. Each automorphism corresponds up to conjugacy to associating a sequence of non-negative, relatively prime integers s_i to the nodes of the extended affine Dynkin diagram corresponding to the simple Lie algebra in the case of an inner automorphism or to the nodes of a twisted affine Dynkin diagram in the case of an outer automorphism. Two such automorphisms are conjugate if and only if the associated integers s_i can be transformed into each other by an automorphism of the affine Dynkin diagram. Moreover, the fixed-point Lie subalgebra can be read off by considering the subdiagram corresponding to the vanishing s_i plus an abelian part (see Proposition 8.6 in [Kac90]).

It is even possible to state when two inner automorphisms are not just conjugate but conjugate under an inner automorphism. This is the case if and only if the associated integers s_i can be transformed into each other by an element in a certain subgroup of the automorphism group of the extended affine Dynkin diagram.

Two outer automorphisms which are conjugate are always conjugate under an inner automorphism, except for the simple type D_4 (see Lemma 7.4).

It is easy to extend Kac's theory to semisimple or even reductive Lie algebras.

2. ORBIFOLD CONSTRUCTION

In [EMS15] (see also [Mö16]) we develop a cyclic orbifold theory for holomorphic vertex operator algebras. Here we give a short summary. Let V be a tame, holomorphic vertex operator algebra (of central charge $c \in 8\mathbb{Z}_{>0}$) and $G = \langle \sigma \rangle$ a finite, cyclic group of automorphisms of V of order $n \in \mathbb{Z}_{>0}$.

By [DLM00], there is an up to isomorphism unique σ^i -twisted V -module, which we call $V(\sigma^i)$, for each $i \in \mathbb{Z}_n$. Moreover, there is a representation $\phi_i : G \rightarrow \text{Aut}_{\mathbb{C}}(V(\sigma^i))$ of G on the vector space $V(\sigma^i)$ such that

$$\phi_i(\sigma) Y_{V(\sigma^i)}(v, x) \phi_i^{-1}(\sigma) = Y_{V(\sigma^i)}(\sigma v, x)$$

²Note that we use the modern sign convention in the definition of twisted modules (see for example [DLM00] as opposed to [Li96]). This is why we included a minus sign in the definition of $\sigma_v = e^{-(2\pi i)v_0}$.

for all $i \in \mathbb{Z}_n$, $v \in V$. This representation is unique up to an n -th root of unity. Let ξ_n be the n -th root of unity $\xi_n := e^{(2\pi i)/n}$. We denote the eigenspace of ϕ_i in $V(\sigma^i)$ corresponding to the eigenvalue ξ_n^j by $W^{(i,j)}$. On $V(\sigma^0) = V$ a possible choice for ϕ_0 is $\phi_0(\sigma) = \sigma$.

The fixed-point vertex operator subalgebra $V^\sigma = W^{(0,0)}$ is tame by [DM97, Miy15, CM16] and has exactly n^2 irreducible modules, namely the $W^{(i,j)}$, $i, j \in \mathbb{Z}_n$ [MT04]. We further show that the conformal weight of $V(\sigma)$ is in $(1/n^2)\mathbb{Z}$, and we define the type $n\{t\}$ of σ where $t \in \mathbb{Z}_n$ such that $t = n^2\rho(V(\sigma)) \pmod{n}$.

Let us assume in the following that σ has type $n\{0\}$, i.e. that $\rho(V(\sigma)) \in (1/n)\mathbb{Z}$. In this case it is possible to choose the representations ϕ_i such that the conformal weights obey

$$\rho(W^{(i,j)}) \in \frac{ij}{n} + \mathbb{Z}$$

and V^σ has fusion rules

$$W^{(i,j)} \boxtimes W^{(l,k)} \cong W^{(i+l, j+k)}$$

for all $i, j, k, l \in \mathbb{Z}_n$ (see [EMS15], Section 5), i.e. the fusion ring of V^σ is the group ring $\mathbb{C}[\mathbb{Z}_n \times \mathbb{Z}_n]$. In particular, all V^σ -modules are simple currents.

We also show that the characters of the irreducible V^σ -modules

$$\text{ch}_{W^{(i,j)}}(\tau) = \text{tr}_{W^{(i,j)}} q^{L_0 - c/24},$$

$i, j \in \mathbb{Z}_n$, $q = e^{(2\pi i)\tau}$, τ a complex variable in the upper half-plane \mathbb{H} , are modular forms of weight 0 for $\Gamma(n)$ if $24 \mid c$ and for $\Gamma(\text{lcm}(3, n))$ otherwise. They even form a vector-valued modular form for the Weil representation of $\text{SL}_2(\mathbb{Z})$ (times a simple character, which is trivial for $24 \mid c$).

In general, a simple vertex operator algebra V is said to satisfy the *positivity assumption* if the conformal weight $\rho(W) > 0$ for any irreducible V -module $W \not\cong V$ and $\rho(V) = 0$.

Now, if V^σ satisfies the positivity assumption, then the direct sum of V^σ -modules

$$(6) \quad V^{\text{orb}(\sigma)} := \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$$

admits the structure of a tame, holomorphic vertex operator algebra of the same central charge as V and is called *orbifold* of V with respect to σ [EMS15]. Note that $\bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)}$ is just the old vertex operator algebra V .

2.1. Inverse Orbifold. The inverse orbifold was first described in [EMS15]. Suppose that the tame, holomorphic vertex operator algebra $V^{\text{orb}(\sigma)}$ is obtained as an orbifold as described above. Then via $\zeta v := \xi_n^i v$ for $v \in W^{(i,0)}$ we define an automorphism ζ of $V^{\text{orb}(\sigma)}$ of order n , and the unique irreducible ζ^j -twisted $V^{\text{orb}(\sigma)}$ -module is given by $V^{\text{orb}(\sigma)}(\zeta^j) = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,j)}$, $j \in \mathbb{Z}_n$ (see [Mö16], Theorem 4.9.6, for the proof). Then

$$(V^{\text{orb}(\sigma)})^{\text{orb}(\zeta)} = \bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)} = V,$$

i.e. orbifolding with ζ is inverse to orbifolding with σ .

3. DIMENSION FORMULAE

The modular properties of the characters of the irreducible V^σ -modules can be used to derive a formula for the dimension of the weight-one space $V_1^{\text{orb}(\sigma)}$ if $c = 24$ and if n is such that the modular curve $X_0(n) = \Gamma_0(n) \backslash \mathbb{H}^*$ has genus zero (where $\mathbb{H}^* = \mathbb{H} \cup P$ with the complex upper half-plane \mathbb{H} and the cusps $P = \mathbb{Q} \cup \{\infty\}$), in which case $\Gamma_0(n)$ is called a genus zero group. The group $\Gamma_0(n)$ has genus zero if and only if $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25$. Then, in particular, the

field of modular functions, i.e. the meromorphic modular forms of weight 0, for the congruence subgroup $\Gamma_0(n)$ is generated by a single meromorphic function t_n called a *Hauptmodul*. This means that every modular function is a rational function of this Hauptmodul, i.e. lies in $\mathbb{C}(t_n)$.

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. The main result of this section is the following theorem:

Theorem 3.1 (Dimension Formula). *Let V be a tame, holomorphic vertex operator algebra of central charge 24 and σ an automorphism of V of type $n\{0\}$, and assume that V^σ satisfies the positivity assumption. Let $V^{\mathrm{orb}(\sigma)} = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$ be the orbifold vertex operator algebra (6). Then for those n with $\Gamma_0(n)$ of genus zero the dimension formula*

$$\sum_{d|n} \frac{\phi((d, n/d))}{(d, n/d)} \left(24 + \frac{n}{d} \dim(V_1^\sigma) - \dim(V_1^{\mathrm{orb}(\sigma^d)}) \right) = 24 + R$$

holds where ϕ is Euler's totient function and

$$R = \sum_{\substack{i \in \mathbb{Z}_n \setminus \{0\} \\ j \in \mathbb{Z}_n}} \sum_{k=0}^{n-1} c_{i,j,k} \mathrm{tr}_{V(\sigma^i)_{k/n}} \phi_i(\sigma^j),$$

$c_{i,j,k} \in \mathbb{Q}$, is a certain linear combination of the traces $\mathrm{tr}_{V(\sigma^i)_{k/n}} \phi_i(\sigma^j)$ on the weight spaces $V(\sigma^i)_{k/n}$ with $i \in \mathbb{Z}_n \setminus \{0\}$, $j \in \mathbb{Z}_n$ and $0 \leq k/n < 1$. For instance for $n = 2, 3, 5, 7, 13$

$$R = \frac{24}{n-1} \sum_{k=1}^{n-1} d_k \sum_{i \in \mathbb{Z}_n \setminus \{0\}} \dim(V(\sigma^i)_{k/n})$$

with $d_k = \sigma_1(n-k)$ where σ_1 is the usual sum-of-divisors function, and for $n = 4$

$$R = 12 \sum_{k=1}^{n-1} d_k \sum_{i \in \mathbb{Z}_n \setminus \{0\}} \dim(V(\sigma^i)_{k/n})$$

with $d_1 = 4$, $d_2 = 2$ and $d_3 = 1$.

Note that if σ is of type $n\{0\}$, then for any $d \mid n$ the automorphism σ^d has type $(n/d)\{0\}$ so that the orbifold vertex operator algebra $V^{\mathrm{orb}(\sigma^d)}$ exists.

Remark 3.2. The sum of the coefficients of $\dim(V_1^\sigma)$ on the left-hand side is given by the the sum of all cusp widths for $\Gamma_0(n)$, which equals the Dedekind psi function

$$\psi(n) = \sum_{d|n} \frac{n}{d} \frac{\phi((d, n/d))}{(d, n/d)} = n \prod_{\substack{p|n \\ p \text{ prime}}} (1 + 1/p)$$

and, since $\Gamma_0(n)$ contains the negative of the identity matrix, is also equal to the index $[\Gamma : \Gamma_0(n)]$.

The dimension formula for $n = 3$ is given in equation (73) of [Mon94] (see also [LS16c], Theorem 4.8). For $n = 2, 3, 5, 7, 13$ a proof is given in [Mö16], Proposition 4.10.4. A complete proof of the theorem will be given in Section 3.1.

The theorem simplifies if the conformal weights satisfy $\rho(V(\sigma^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$ since then $R = 0$.

Corollary 3.3. *In the situation of the above theorem let us assume that the conformal weights of the twisted modules obey $\rho(V(\sigma^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$. Then:*

- For $n = 2, 3, 5, 7, 13$,

$$\dim(V_1^{\mathrm{orb}(\sigma)}) = 24 + (n+1) \dim(V_1^\sigma) - \dim(V_1).$$

- For $n = 4$,

$$\dim(V_1^{\text{orb}(\sigma)}) = 24 + 6 \dim(V_1^\sigma) - \frac{3}{2} \dim(V_1^{\sigma^2}) - \frac{1}{2} \dim(V_1).$$

- For $n = 6$,

$$\begin{aligned} \dim(V_1^{\text{orb}(\sigma)}) \\ = 24 + 12 \dim(V_1^\sigma) - 4 \dim(V_1^{\sigma^2}) - 3 \dim(V_1^{\sigma^3}) + \dim(V_1). \end{aligned}$$

- For $n = 8$,

$$\begin{aligned} \dim(V_1^{\text{orb}(\sigma)}) \\ = 24 + 12 \dim(V_1^\sigma) - 3 \dim(V_1^{\sigma^2}) - \frac{3}{4} \dim(V_1^{\sigma^4}) - \frac{1}{4} \dim(V_1). \end{aligned}$$

Proof. In the prime order case the result is immediate. For $n = 4$ we obtain

$$\dim(V_1) + \frac{1}{2} \dim(V_1^{\text{orb}(\sigma^2)}) + \dim(V_1^{\text{orb}(\sigma)}) = 36 + 6 \dim(V_1^\sigma)$$

for σ and

$$\dim(V_1) + \dim(V_1^{\text{orb}(\sigma^2)}) = 24 + 3 \dim(V_1^{\sigma^2})$$

for σ^2 of order 2, which combined gives the result for $n = 4$. For $n = 6, 8$ we proceed in the same manner. \square

Of course, analogous formulae can be written down for any n such that $\Gamma_0(n)$ is of genus zero.

3.1. Proof of Dimension Formula. In this section we prove Theorem 3.1. Let V be a tame, holomorphic vertex operator algebra of central charge 24 and σ an automorphism of V of type $n\{0\}$, and assume that V^σ satisfies the positivity assumption. Let $V^{\text{orb}(\sigma)} = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$ be the orbifold vertex operator algebra (6).

Then, as shown in [EMS15], the characters $\text{ch}_{W^{(i,j)}}(\tau)$ transform with weight 0 under the Weil representation of the group $D = \mathbb{Z}_n \times \mathbb{Z}_n$ with quadratic form $q((i,j)) = ij/n + \mathbb{Z}$, i.e.

$$\begin{aligned} \text{ch}_{W^{(i,j)}}(T.\tau) &= e^{(2\pi i)ij/n} \text{ch}_{W^{(i,j)}}(\tau), \\ \text{ch}_{W^{(i,j)}}(S.\tau) &= \frac{1}{n} \sum_{k,l \in \mathbb{Z}_n} e^{-(2\pi i)(il+jk)/n} \text{ch}_{W^{(k,l)}}(\tau) \end{aligned}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the standard generators of $\Gamma = \text{SL}_2(\mathbb{Z})$. This implies that

$$\text{ch}_{W^{(0,0)}}(\tau) = \text{ch}_{V^\sigma}(\tau)$$

is invariant under $\Gamma_0(n)$ [Sch09].

The cusps $P = \mathbb{Q} \cup \{\infty\}$ fall into $\sum_{c|n} \phi((c, n/c))$ classes modulo $\Gamma_0(n)$. The cusp classes can be represented by the rational numbers a/c with $(a, c) = 1$ where c ranges over the positive divisors of n and a over a complete set of representatives of $\mathbb{Z}_{(c, n/c)}^\times$. We denote by $t_s := (n/c)/(c, n/c)$ the width of the cusp $s = a/c$.

For a cusp $s \in P = \mathbb{Q} \cup \{\infty\}$ we define

$$F_s(\tau) := \sum_{\substack{M \in \Gamma_0(n) \setminus \Gamma \\ M.\infty = s}} \text{ch}_{W^{(0,0)}}(M.\tau).$$

Lemma 3.4. *Let $s = a/c \in \mathbb{Q}$ be a cusp of $\Gamma_0(n)$ with $(a, c) = 1$ and $c \mid n$. Choose d such that $ad = 1 \pmod{c}$. Then*

$$F_s(\tau) = \frac{1}{(c, n/c)} \sum_{\substack{i, j \in \mathbb{Z}_{n/c} \\ ij \equiv 0 \pmod{t_s}}} e^{-(2\pi i)(dij/t_s)/(c, n/c)} \text{ch}_{W^{(ci, cj)}}(\tau).$$

Note that ij/t_s is well-defined modulo $(c, n/c)$.

Proof. Choose a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Then the cosets in $\Gamma_0(n) \backslash \Gamma$ sending ∞ to s are given by MT^j where j ranges over a complete set of residues modulo t_s . The statement now follows from the formula [Sch09, Sch15]

$$(7) \quad \text{ch}_{W^{(0,0)}}(M.\tau) = \frac{c}{n} \sum_{i, j \in \mathbb{Z}_{n/c}} e^{-(2\pi i)dci/n} \text{ch}_{W^{(ci, cj)}}(\tau).$$

□

The orbifold vertex operator algebra for σ^d where $d \mid n$ is given by

$$V^{\text{orb}(\sigma^d)} = \bigoplus_{\substack{j \in \mathbb{Z}_{n/d} \\ k \in \mathbb{Z}_d}} W^{(dj, kn/d)}.$$

Together with the above lemma this implies the following relation between the characters of the orbifolds $V^{\text{orb}(\sigma^d)}$ and the functions F_s .

Proposition 3.5. *The following formula holds:*

$$\sum_{s \in \Gamma_0(n) \backslash P} F_s(\tau) = \sum_{d \mid n} \frac{\phi((d, n/d))}{(d, n/d)} \text{ch}_{V^{\text{orb}(\sigma^d)}}(\tau).$$

So far the results do not depend on the genus of $\Gamma_0(n)$.

Lemma 3.6. *Suppose $\Gamma_0(n)$ has genus zero, and let $s \in P$ be a cusp. If s is not equivalent to ∞ under $\Gamma_0(n)$, then there is a modular function f_s for $\Gamma_0(n)$ which is holomorphic and non-zero on \mathbb{H}^* except for $\text{ord}_s(f_s) = -1/t_s$ and $\text{ord}_\infty(f_s) = 1$.*

If s is equivalent to ∞ , there is a modular function f_s for $\Gamma_0(n)$ which is holomorphic and non-zero on \mathbb{H}^ except for $\text{ord}_s(f_s) = -1 = -1/t_s$ and $\text{ord}_0(f_s) = 1/n$.*

The functions f_s are unique up to a factor.

Proof. The modular curve $X_0(n) = \Gamma_0(n) \backslash \mathbb{H}^*$ has genus zero if and only if every divisor is a principal divisor. Hence, any divisor D satisfying

$$\sum_{s \in \Gamma_0(n) \backslash P} \text{ord}_s(D)t_s = 0$$

is the divisor of a meromorphic function on $X_0(n)$ that is holomorphic and non-zero away from the cusps. □

Lemma 3.7. *Suppose $\Gamma_0(n)$ has genus zero. Then the character $\text{ch}_{V^\sigma}(\tau)$ can be written as*

$$\text{ch}_{V^\sigma}(\tau) = \sum_{s \in \Gamma_0(n) \backslash P} g_s(f_s(\tau)) + c$$

with complex polynomials g_s and a complex constant c . The polynomials g_s are unique up to constants (once we fix a choice of the f_s).

Proof. The character $\text{ch}_{V^\sigma}(\tau)$ is a modular form of weight 0 which is holomorphic on the upper half-plane and has poles at the cusps. Since f_s has its only pole at the cusp s there are complex polynomials g_s such that

$$\text{ch}_{V^\sigma}(\tau) - \sum_{s \in \Gamma_0(n) \setminus P} g_s(f_s(\tau))$$

is holomorphic on the upper half-plane and at all cusps and hence constant. \square

For definiteness we choose the polynomials g_s to have no constant term. We can make the statement of the above theorem even more precise. Noting that (7) implies that $\text{ch}_{V^\sigma}(\tau)$ has a pole of order -1 at every cusp s for $\Gamma_0(n)$ and comparing this with the pole of order $-1/t_s$ of the function f_s at the cusp s , where t_s is the width of the cusp s , we see that

$$\text{ch}_{V^\sigma}(\tau) = c + \sum_{s \in \Gamma_0(n) \setminus P} \sum_{i=1}^{t_s} c_{s,i} f_s(\tau)^i$$

for certain constants $c_{s,i} \in \mathbb{C}$, $s \in \Gamma_0(n) \setminus P$, $i \in \{1, \dots, t_s\}$, i.e. the polynomial

$$g_s(x) = \sum_{i=1}^{t_s} c_{s,i} x^i$$

is of degree t_s with no constant term. The constant c is given by $\dim(V_1^\sigma) - [f_\infty](0)$ where $[f_\infty](0)$ denotes the constant coefficient in the q -expansion of f_∞ since all $f_s(\tau)$ vanish at ∞ except for f_∞ .

Suppose we have determined the polynomials g_s or equivalently the constants $c_{s,i}$. Then we can calculate the expansions of $\text{ch}_{V^\sigma}(\tau)$ at the different cusps, and using Proposition 3.5 we can then compute

$$\sum_{d|n} \frac{\phi((d, n/d))}{(d, n/d)} \dim(V_1^{\text{orb}(\sigma^d)}) = \sum_{s \in \Gamma_0(n) \setminus P} [F_s](0).$$

There are several ways to determine the coefficients of the polynomials g_s . One possibility is as follows. Note that

$$\text{ch}_{V^\sigma}(\tau) = \text{ch}_{W(0,0)}(\tau) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} T(\mathbf{1}, 0, j, \tau)$$

in terms of twisted characters (see e.g. [EMS15] for their definition and properties in the orbifold context) so that

$$\begin{aligned} \text{ch}_{V^\sigma}(M, \tau) &= \frac{1}{n} \sum_{j \in \mathbb{Z}_n} T(\mathbf{1}, (0, j)M, \tau) \\ &= \frac{1}{n} \sum_{j \in \mathbb{Z}_n} T(\mathbf{1}, cj, dj, \tau) \\ &= \frac{1}{n} \sum_{j \in \mathbb{Z}_n} \text{tr}_{V(\sigma^{cj})} \phi_{cj}(\sigma^{dj}) q^{L_0-1} \end{aligned}$$

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Hence we can determine the coefficients of the polynomials g_s in the decomposition

$$\text{ch}_{V^\sigma}(\tau) = \sum_{s \in \Gamma_0(n) \setminus P} g_s(f_s(\tau)) + c$$

by expanding both sides at the cusp s using the above formula for $\text{ch}_{V^\sigma}(M, \tau)$ with $M \cdot \infty = s$ and comparing the t_s many singular q -coefficients.

We then obtain

$$\sum_{d|n} \frac{\phi((d, n/d))}{(d, n/d)} \dim(V_1^{\text{orb}(\sigma^d)}) = \sum_{s \in \Gamma_0(n) \backslash P} [F_s](0)$$

explicitly as some constant plus a multiple of $\dim(V_1^\sigma)$ plus some linear combination of $\text{tr}_{V(\sigma^i)_{k/n}} \phi_i(\sigma^j)$ with $k/n < 1$.

By definition, F_s is a sum of $|\{M \in \Gamma_0(n) \backslash \Gamma \mid M.\infty = s\}| = t_s$ many terms, each containing $\dim(V_1^\sigma)$ once, so that every F_s contributes $t_s \dim(V_1^\sigma)$, and hence this dimension appears with a total multiplicity of $\sum_{s \in \Gamma_0(n) \backslash P} t_s = [\Gamma : \Gamma_0(n)]$ (see Remark 3.2).

Finally, we describe our choice of the functions f_s for some of the cases considered in Theorem 3.1:

- First, let $n = 2, 3, 5, 7, 13$. Then a Hauptmodul for $\Gamma_0(n)$ is given by the eta product

$$t_n(\tau) = \left(\frac{\eta(\tau)}{\eta(n\tau)} \right)^{24/(n-1)} = q^{-1} - \frac{24}{n-1} + \dots$$

where $\eta(\tau)$ is the Dedekind eta function. The Hauptmodul is holomorphic and non-zero on \mathbb{H} . There are two classes of cusps for $\Gamma_0(n)$, represented by $1/n (= \infty)$ and $1/1 (= 0)$. We define the functions

$$f_{1/n}(\tau) := t_n(\tau) = \left(\frac{\eta(\tau)}{\eta(n\tau)} \right)^{24/(n-1)},$$

$$f_{1/1}(\tau) := \frac{1}{t_n(\tau)} = \left(\frac{\eta(n\tau)}{\eta(\tau)} \right)^{24/(n-1)}$$

for the cusps $1/n$ and $1/1$ for $\Gamma_0(n)$, respectively. These functions are holomorphic and non-zero on the upper half-plane \mathbb{H} and have exactly the poles and zeroes at the cusps demanded in Lemma 3.6.

- Now suppose $n = 4$. A Hauptmodul for $\Gamma_0(4)$ is given by

$$t_4(\tau) = \frac{\eta(\tau)^8}{\eta(4\tau)^8} = q^{-1} - 8 + \dots$$

There are three classes of cusps for $\Gamma_0(4)$, represented by $1/4 (= \infty)$, $1/2$ and $1/1 (= 0)$. We define the functions

$$f_{1/4}(\tau) := t_4(\tau) = \frac{\eta(\tau)^8}{\eta(4\tau)^8},$$

$$f_{1/2}(\tau) := \frac{1}{t_4(\tau) + 16} = \frac{\eta(\tau)^8 \eta(4\tau)^{16}}{\eta(2\tau)^{24}},$$

$$f_{1/1}(\tau) := \frac{1}{t_4(\tau)} = \frac{\eta(4\tau)^8}{\eta(\tau)^8}$$

for the cusps $1/4$, $1/2$ and $1/1$ for $\Gamma_0(4)$, respectively. These functions are holomorphic and non-zero on the upper half-plane \mathbb{H} and have exactly the poles and zeroes at the cusps demanded in Lemma 3.6.

- Let $n = 6$. A Hauptmodul for $\Gamma_0(6)$ is given by

$$t_6(\tau) = \frac{\eta(\tau)^5 \eta(3\tau)}{\eta(2\tau) \eta(6\tau)^5} = q^{-1} - 5 + \dots$$

There are four classes of cusps for $\Gamma_0(6)$, represented by $1/6$ ($= \infty$), $1/3$, $1/2$ and $1/1$ ($= 0$). We define the functions

$$\begin{aligned} f_{1/6}(\tau) &:= t_6(\tau) = \frac{\eta(\tau)^5 \eta(3\tau)}{\eta(2\tau) \eta(6\tau)^5}, \\ f_{1/3}(\tau) &:= \frac{1}{t_6(\tau) + 8} = \frac{\eta(\tau)^3 \eta(6\tau)^9}{\eta(2\tau)^3 \eta(3\tau)^9}, \\ f_{1/2}(\tau) &:= \frac{1}{t_6(\tau) + 9} = \frac{\eta(\tau)^4 \eta(6\tau)^8}{\eta(2\tau)^8 \eta(3\tau)^4}, \\ f_{1/1}(\tau) &:= \frac{1}{t_6(\tau)} = \frac{\eta(2\tau) \eta(6\tau)^5}{\eta(\tau)^5 \eta(3\tau)} \end{aligned}$$

for the cusps $1/6$, $1/3$, $1/2$ and $1/1$ for $\Gamma_0(6)$, respectively. These functions are holomorphic and non-zero on the upper half-plane \mathbb{H} and have exactly the poles and zeroes at the cusps demanded in Lemma 3.6.

- Let $n = 8$. A Hauptmodul for $\Gamma_0(8)$ is given by

$$t_8(\tau) = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4} = q^{-1} - 4 + \dots$$

There are four classes of cusps for $\Gamma_0(8)$, represented by $1/8$ ($= \infty$), $1/4$, $1/2$ and $1/1$ ($= 0$). We define the functions

$$\begin{aligned} f_{1/8}(\tau) &:= t_8(\tau) = \frac{\eta(\tau)^4 \eta(4\tau)^2}{\eta(2\tau)^2 \eta(8\tau)^4}, \\ f_{1/4}(\tau) &:= \frac{1}{t_8(\tau) + 4} = \frac{\eta(2\tau)^4 \eta(8\tau)^8}{\eta(4\tau)^{12}}, \\ f_{1/2}(\tau) &:= \frac{1}{t_8(\tau) + 8} = \frac{\eta(\tau)^4 \eta(4\tau)^2 \eta(8\tau)^4}{\eta(2\tau)^{10}}, \\ f_{1/1}(\tau) &:= \frac{1}{t_8(\tau)} = \frac{\eta(2\tau)^2 \eta(8\tau)^4}{\eta(\tau)^4 \eta(4\tau)^2} \end{aligned}$$

for the cusps $1/8$, $1/4$, $1/2$ and $1/1$ for $\Gamma_0(8)$, respectively. These functions are holomorphic on the upper half-plane \mathbb{H} and have exactly the poles and zeroes at the cusps demanded in Lemma 3.6.

4. UNIQUENESS PROCEDURE

The following idea is due to Lam and Shimakura (see [LS16c], Theorem 5.2) and makes use of the inverse orbifold construction developed in [EMS15].

4.1. General Idea. Let U be a tame, holomorphic vertex operator algebra of central charge c and g an automorphism of U of type $n\{0\}$ for some $n \in \mathbb{Z}_{>0}$. Assume that U^g satisfies the positivity assumption. Then by [EMS15] we can apply the \mathbb{Z}_n -orbifold construction to U and g and obtain the orbifolded vertex operator algebra $U^{\text{orb}(g)}$, which is also tame, holomorphic and of central charge c .

On the other hand, let V be a tame, holomorphic vertex operator algebra of central charge c with weight-one Lie algebra $V_1 \cong U_1^{\text{orb}(g)}$. Suppose that there exists an element $v \in V_1$ such that v_0 acts semisimply on V , the inner automorphism $\sigma_v := e^{-(2\pi i)v_0}$ has type $n\{0\}$ on V and V^{σ_v} satisfies the positivity assumption. Then we can apply the \mathbb{Z}_n -orbifold construction to V and σ_v and obtain the tame, holomorphic vertex operator algebra $V^{\text{orb}(\sigma_v)}$ of central charge c . Assume that $V^{\text{orb}(\sigma_v)} \cong U$.

By the inverse orbifold construction [EMS15] there exists an automorphism $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_v)})$ of type $n\{0\}$ with $(V^{\text{orb}(\sigma_v)})^\zeta = V^{\sigma_v}$ such that the orbifold

construction with $V^{\text{orb}(\sigma_v)}$ and ζ gives back V , i.e. $(V^{\text{orb}(\sigma_v)})^{\text{orb}(\zeta)} = V$. Finally, assume that under the isomorphism $V^{\text{orb}(\sigma_v)} \cong U$ the inverse orbifold automorphism $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_v)})$ of σ_v is conjugate to $g \in \text{Aut}(U)$ (and in particular $V_1^{\sigma_v} \cong U_1^g$). Then the orbifold of $V^{\text{orb}(\sigma_v)}$ by ζ , which is V , is isomorphic to the orbifold $U^{\text{orb}(g)}$ of U by g , i.e. $V \cong U^{\text{orb}(g)}$.

4.2. Concrete Approach. In order to prove the uniqueness conjecture (Conjecture 1.3) for some cases on Schellekens' list we specialise to the following situation.

Part 1: Construction. Let $U := V_L$ be the lattice vertex operator algebra associated with some Niemeier lattice L . Then V_L is tame, holomorphic and of central charge 24. Let $\nu \in \text{Aut}(L)$ be an automorphism of the lattice L and $\hat{\nu} \in \text{Aut}(V_L)$ a standard lift of ν . We denote by $n \in \mathbb{Z}_{>0}$ the order of $\hat{\nu}$, which is $\text{ord}(\nu)$ or $2 \text{ord}(\nu)$ (possible order doubling). Suppose that the conformal weight $\rho(V_L(\hat{\nu})) = 0 \pmod{1/n}$, i.e. $\hat{\nu}$ is of type $n\{0\}$. Then the fixed-point vertex operator subalgebra $V_L^{\hat{\nu}}$ satisfies the positivity assumption. We apply the orbifold construction [EMS15] and obtain the orbifolded vertex operator algebra $V_L^{\text{orb}(\hat{\nu})}$, which is again tame, holomorphic and of central charge 24. Then we compute the dimension of the Lie algebra $(V_L^{\text{orb}(\hat{\nu})})_1$, which has to be one of the 71 Lie algebras on Schellekens' list. Suppose we are able to determine the Lie algebra structure of $(V_L^{\text{orb}(\hat{\nu})})_1$ up to isomorphism.

Part 2: Inner Automorphism. We want to show that $V_L^{\text{orb}(\hat{\nu})}$ is up to isomorphism the only tame, holomorphic vertex operator algebra of central charge 24 with weight-one Lie algebra isomorphic to $(V_L^{\text{orb}(\hat{\nu})})_1$. To this end, consider another tame, holomorphic vertex operator algebra V of central charge 24 with $V_1 \cong (V_L^{\text{orb}(\hat{\nu})})_1$.

The Lie algebra $V_1 \cong (V_L^{\text{orb}(\hat{\nu})})_1$ is reductive or more precisely semisimple, abelian or zero. Assume that V_1 is semisimple. We fix a Cartan subalgebra \mathfrak{h} of V_1 . Let $h \in \mathfrak{h}$ such that h_0 acts semisimply on V .³ The action of h_0 on V_1 is given by ad_h . In particular, $h_0 x = [h, x] = \alpha(h)x$ for x in the root space of V_1 associated with the root $\alpha \in \Phi \subseteq \mathfrak{h}^*$.

Let us consider the inner automorphism $\sigma_h := e^{-(2\pi i)h_0}$ on V and assume it has type $n\{0\}$, that V^{σ_h} satisfies the positivity assumption and that $V_1^{\sigma_h} \cong (V_L^{\hat{\nu}})_1$. Then we can apply the orbifold construction and obtain the tame, holomorphic vertex operator algebra $V^{\text{orb}(\sigma_h)}$ of central charge 24. Assume that we can compute the dimension $d := \dim(V_1^{\text{orb}(\sigma_h)})$. More specifically, we assume that $\rho(V(\sigma_h^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$ so that we can apply the dimension formula in Corollary 3.3. Then suppose that $d = \dim((V_L)_1)$.

We know that $(V_L^{\hat{\nu}})_1$ is a fixed-point Lie subalgebra of $(V_L)_1$ under an automorphism of order dividing n , namely under $\hat{\nu}|_{(V_L)_1}$. Then, so is $V_1^{\sigma_h} \cong (V_L^{\hat{\nu}})_1$. Suppose that $(V_L)_1$ is the only Lie algebra on Schellekens' list in dimension d with $(V_L^{\hat{\nu}})_1$ as a possible fixed-point Lie subalgebra under an automorphism of order dividing n (the possible fixed-point Lie subalgebras can be determined using Kac's theory as described in Section 1.7). Then we know that $V_1^{\text{orb}(\sigma_h)} \cong (V_L)_1$ since $V_1^{\text{orb}(\sigma_h)}$ has dimension d by definition and has $V_1^{\sigma_h} \cong (V_L^{\hat{\nu}})_1$ as fixed-point Lie subalgebra (under the inverse orbifold automorphism ζ restricted to $V_1^{\text{orb}(\sigma_h)}$, see below). Then, finally $V^{\text{orb}(\sigma_h)} \cong V_L$ since it is shown in [DM04b] that a tame, holomorphic vertex operator algebra of central charge 24 with a Lie algebra of rank 24 is already isomorphic to a lattice vertex operator algebra.

³It is shown in [DM04b] and [Mas14], Section 3.3, that every V -module is a completely reducible V_1 -module. Then if $v \in V_1$ is a semisimple element, the zeroth mode v_0 acts semisimply on every V -module (and in particular on V itself).

Part 3: Unique Conjugacy Class. Let $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_h)})$ be the inverse orbifold automorphism such that $(V^{\text{orb}(\sigma_h)})^{\text{orb}(\zeta)} \cong V$. Assume that we can show that ζ is conjugate to $\hat{\nu} \in \text{Aut}(V_L)$ under the isomorphism $V^{\text{orb}(\sigma_h)} \cong V_L$. More specifically we will show that there is a unique conjugacy class of automorphisms g of V_L of order n with fixed-point Lie subalgebra $(V_L^g)_1$ isomorphic to $V_1^{\sigma_h} \cong (V_L^{\hat{\nu}})_1$ and possibly the condition that the conformal weight $\rho(V_L(g)) = 1$.

Knowing that ζ and $\hat{\nu}$ are conjugate we conclude, as described above, that $V \cong V_L^{\text{orb}(\hat{\nu})}$. This then proves that the vertex operator algebra $V_L^{\text{orb}(\hat{\nu})}$ is the up to isomorphism unique tame, holomorphic vertex operator algebra of central charge 24 with weight-one space isomorphic as Lie algebra to $(V_L^{\text{orb}(\hat{\nu})})_1$.

5. PART 1: CONSTRUCTION

The construction of orbifolds $V_L^{\text{orb}(\hat{\nu})}$ associated with a holomorphic lattice vertex operator algebra V_L and an automorphism $\hat{\nu} \in \text{Aut}(V_L)$ of type $n\{0\}$ obtained as a standard lift of a lattice automorphism $\nu \in \text{Aut}(L)$ is described in [EMS15], Section 8 and [Mö16], Sections 5.6, 6.2 and 6.3. Applying these methods we obtain orbifold constructions of the following tame, holomorphic vertex operator algebras of central charge 24:

Proposition 5.1 (Part 1). *For each entry of the following table there exists an automorphism ν of the Niemeier lattice L of order n and cycle shape s such that the orbifold vertex operator algebra $V_L^{\text{orb}(\hat{\nu})}$ of V_L by a standard lift $\hat{\nu}$ (also of order n) has weight-one Lie algebra $(V_L^{\text{orb}(\hat{\nu})})_1$ as indicated.*

No.	L	n	s	$(V_L^{\hat{\nu}})_1$	$(V_L^{\text{orb}(\hat{\nu})})_1$	[Sch93]
(1)	$N(A_9^2 D_6)$	2	$1^{-8} 2^{16}$	$A_5 C_5 D_5 \mathbb{C}$	$A_5 C_5 E_{6,2}$	44
(2)	$N(A_7^2 D_5^2)$	2	$1^{-8} 2^{16}$	$A_1^2 A_3 A_7 B_2^2$	$A_3 A_{7,2} C_3^2$	33
(3)	$N(A_8^3)$	2	$1^{-8} 2^{16}$	$A_8 B_4$	$A_{8,2} F_{4,2}$	36
(4)	$N(E_8^3)$	2	$1^{-8} 2^{16}$	$D_8 E_8$	$B_8 E_{8,2}$	62
(5)	$N(A_5^4 D_4)$	2	$1^{-8} 2^{16}$	$A_1 A_2^2 A_3 A_5 B_2 \mathbb{C}$	$A_2^2 A_{5,2} B_2$	26
(6)	$N(E_6^4)$	2	$1^{-8} 2^{16}$	$C_4^2 F_4^2$	$C_8 F_4^2$	52
(7)	$N(A_4^6)$	2	$1^{-8} 2^{16}$	$A_4^2 B_2^2$	$A_{4,2}^2 C_{4,2}$	22
(8)	$N(A_2^{12})$	2	$1^{-8} 2^{16}$	$A_1^4 A_2^4$	$A_{2,2}^4 D_{4,4}$	13
(9)	$N(A_{11} D_7 E_6)$	2	$1^{-8} 2^{16}$	$A_1 B_5 D_6 F_4$	$B_5 E_{7,2} F_4$	53
(10)	$N(A_{11} D_7 E_6)$	2	$1^{-8} 2^{16}$	$B_2 B_4 C_4 C_6$	$B_4 C_6^2$	48
(11)	$N(A_4^6)$	5	$1^{-1} 5^5$	$A_4 \mathbb{C}^4$	$A_{4,5}^2$	9
(12a)	$N(D_{10} E_7^2)$	4	$1^2 2^{-9} 4^{10}$	$A_1 A_4 A_7 B_3 \mathbb{C}$	$A_4 A_{9,2} B_3$	40
(12b)	$N(D_{10} E_7^2)$	4	$2^{-4} 4^8$	$A_1 A_4 A_7 B_3 \mathbb{C}$	$A_4 A_{9,2} B_3$	40
(13)	$N(D_{10} E_7^2)$	4	$2^{-4} 4^8$	$A_7 B_2 B_6 \mathbb{C}$	$B_6 C_{10}$	56
(14)	$N(E_6^4)$	6	$1^3 2^{-3} 3^{-9} 6^9$	$A_1 A_2 C_4 \mathbb{C}$	$A_1 C_{5,3} G_{2,2}$	21
(15)	$N(D_4^6)$	8	$4^{-2} 8^4$	$A_3 \mathbb{C}^7$	$A_{1,2} A_{3,4}^3$	7

Note that the automorphisms $\hat{\nu}$ in cases (12a) and (12b) are in fact conjugate in $\text{Aut}(V_L)$ as is shown in Proposition 7.5 while the lattice automorphisms ν they are lifted from cannot be conjugate in $\text{Aut}(L)$ since they do not have the same cycle shape.

Sketch of Proof. For cases (1), (2) and (15) we let ν be the automorphism, unique up to conjugacy, specified by the following conditions:

- (1) There are exactly two conjugacy classes in $\text{Aut}(N(A_9^2 D_6))$ of cycle shape $1^{-8}2^{16}$ and length 113 400. For exactly one of them no order doubling occurs. This is the one we choose as ν .
- (2) There are exactly two conjugacy classes in $\text{Aut}(N(A_7^2 D_5^2))$ of cycle shape $1^{-8}2^{16}$ and length 403 200. For both of them no order doubling occurs. We choose as ν the only one with $(V_L^\nu)_1 \cong A_1^2 A_3 A_7 B_2^2$.
- (15) There are exactly four conjugacy classes in $\text{Aut}(N(D_4^6))$ of cycle shape $4^{-2}8^4$ and length 26 418 075 402 240. Two of them exhibit no order doubling. We choose as ν the only one of those two with $(V_L^\nu)_1 \cong A_3 C^7$.

For the remaining cases we let ν be a representative of the unique conjugacy class

- (3) in $\text{Aut}(N(A_8^3))$ of cycle shape $1^{-8}2^{16}$ and length 1 088 640,
- (4) in $\text{Aut}(N(E_8^3))$ of cycle shape $1^{-8}2^{16}$ and length 2 090 188 800,
- (5) in $\text{Aut}(N(A_5^4 D_4))$ of cycle shape $1^{-8}2^{16}$ and length 518 400,
- (6) in $\text{Aut}(N(E_6^4))$ of cycle shape $1^{-8}2^{16}$ and length 12 150,
- (7) in $\text{Aut}(N(A_4^6))$ of cycle shape $1^{-8}2^{16}$ and length 216 000,
- (8) in $\text{Aut}(N(A_2^{12}))$ of cycle shape $1^{-8}2^{16}$ and length 641 520,
- (9) in $\text{Aut}(N(A_{11} D_7 E_6))$ of cycle shape $1^{-8}2^{16}$ and length 1 575,
- (10) in $\text{Aut}(N(A_{11} D_7 E_6))$ of cycle shape $1^{-8}2^{16}$ and length 218 295,
- (11) in $\text{Aut}(N(A_4^6))$ of cycle shape $1^{-1}5^5$ and length 119 439 360 000,
- (12a) in $\text{Aut}(N(D_{10} E_7^2))$ of cycle shape $1^2 2^{-9} 4^{10}$ and length 877 879 296 000,
- (12b) in $\text{Aut}(N(D_{10} E_7^2))$ of cycle shape $2^{-4} 4^8$ and length 1 316 818 944 000,
- (13) in $\text{Aut}(N(D_{10} E_7^2))$ of cycle shape $2^{-4} 4^8$ and length 18 289 152 000,
- (14) in $\text{Aut}(N(E_6^4))$ of cycle shape $1^3 2^{-3} 3^{-9} 6^9$ and length 1 719 926 784 000,

respectively. For each case (1) to (15) we let $\hat{\nu}$ be a standard lift of ν . No order doubling occurs, and the automorphism $\hat{\nu}$ of V_L is of type $n\{0\}$.

We observe that the conformal weights of the twisted modules satisfy $\rho(V(\hat{\nu}^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$. Hence the dimension of the weight-one Lie algebra $(V_L^{\text{orb}(\hat{\nu})})_1$ can be computed using Corollary 3.3. (Alternatively, this dimension can be obtained by the method described in Section 8 of [EMS15]).

For each case (1) to (15) we know $\dim((V_L^{\text{orb}(\hat{\nu})})_1)$, and we know that $(V_L^{\text{orb}(\hat{\nu})})_1$ contains $(V_L^\nu)_1$ as the fixed-point Lie subalgebra under an automorphism of order dividing n , namely under the inverse orbifold automorphism with respect to $\hat{\nu}$. This is enough to uniquely determine the structure of $(V_L^{\text{orb}(\hat{\nu})})_1$ using Kac's theory (see Section 1.7) and Schellekens' list. Actually, for the case (11) we need some additional arguments, but this case is described in detail in [EMS15]. \square

6. PART 2: INNER AUTOMORPHISM

In the following let V be a tame, holomorphic vertex operator algebra of central charge 24 with semisimple weight-one Lie algebra V_1 . Let \mathfrak{h} be a Cartan subalgebra of V_1 . For each of the fifteen affine structures appearing on the right-hand side of

the table in Proposition 5.1 we fix a semisimple element $h \in \mathfrak{h}$ (so that h_0 acts semisimply on V) as in the following table such that $\sigma_h = e^{-(2\pi i)h_0}$ has order n :

No.	V_1	h	Ord. on V_1	n	$\langle h, h \rangle$	$V_1^{\sigma_h}$
(1)	$A_5 C_5 E_{6,2}$	$(0, 0, \frac{\Lambda_1^\vee + \Lambda_6^\vee}{2})$	$(1, 1, 2)$	2	2	$A_5 C_5 D_5 \mathbb{C}$
(2)	$A_3 A_{7,2} C_3^2$	$(0, 0, \frac{\Lambda_2^\vee}{2}, \frac{\Lambda_2^\vee}{2})$	$(1, 1, 2, 2)$	2	2	$A_1^2 A_3 A_7 B_2^2$
(3)	$A_{8,2} F_{4,2}$	$(0, \frac{\Lambda_4^\vee}{2})$	$(1, 2)$	2	2	$A_8 B_4$
(4)	$B_8 E_{8,2}$	$(\frac{\Lambda_8^\vee}{2}, 0)$	$(2, 1)$	2	2	$D_8 E_8$
(5)	$A_2^2 A_{5,2}^2 B_2$	$(0, 0, \frac{\Lambda_2^\vee + \Lambda_4^\vee}{2}, 0, 0)$	$(1, 1, 2, 1, 1)$	2	2	$A_1 A_2^2 A_3 A_5 B_2 \mathbb{C}$
(6)	$C_8 F_4^2$	$(\frac{\Lambda_4^\vee}{2}, 0, 0)$	$(2, 1, 1)$	2	2	$C_4^2 F_4^2$
(7)	$A_{4,2}^2 C_{4,2}$	$(0, 0, \frac{\Lambda_2^\vee}{2})$	$(1, 1, 2)$	2	2	$A_4^2 B_2^2$
(8)	$A_{2,2}^4 D_{4,4}$	$(0, 0, 0, 0, \frac{\Lambda_2^\vee}{2})$	$(1, 1, 1, 1, 2)$	2	2	$A_1^4 A_2^4$
(9)	$B_5 E_{7,2} F_4$	$(0, \frac{\Lambda_6^\vee}{2}, 0)$	$(1, 2, 1)$	2	2	$A_1 B_5 D_6 F_4$
(10)	$B_4 C_6^2$	$(0, 0, \frac{\Lambda_4^\vee}{2})$	$(1, 1, 2)$	2	2	$B_2 B_4 C_4 C_6$
(11)	$A_{4,5}^2$	$(0, \frac{\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee}{5})$	$(1, 5)$	5	2	$A_4 \mathbb{C}^4$
(12)	$A_4 A_{9,2} B_3$	$(0, \frac{\Lambda_1^\vee + 3\Lambda_3^\vee}{4}, 0)$	$(1, 4, 1)$	4	3	$A_1 A_4 A_7 B_3 \mathbb{C}$
(13)	$B_6 C_{10}$	$(0, \frac{\Lambda_2^\vee}{4} + \frac{\Lambda_{10}^\vee}{2})$	$(1, 4)$	4	2	$A_7 B_2 B_6 \mathbb{C}$
(14)	$A_1 C_{5,3} G_{2,2}$	$(0, \frac{\Lambda_4^\vee}{6} + \frac{2\Lambda_5^\vee}{3}, \frac{\Lambda_3^\vee}{3})$	$(1, 6, 3)$	6	8	$A_1 A_2 C_4 \mathbb{C}$
(15)	$A_{1,2} A_{3,4}^3$	$(\frac{\Lambda_1^\vee}{2}, x, x, 0)$	$(2, 8, 8, 1)$	8	2	$A_3 \mathbb{C}^7$

with $x = \frac{\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_3^\vee}{8}$. We give h in terms of the fundamental coweights (as labelled e.g. in [Hum73]). The fixed-point Lie subalgebra $V_1^{\sigma_h}$ can be easily read off from the definitions of h , using Kac's theory (see Section 1.7), and we observe that $V_1^{\sigma_h} \cong (V_L^\vee)_1$ with V_L^\vee from the previous section. Of course, h is chosen precisely such that this is the case.

Since σ_h has order n , the eigenvalues of h_0 lie in $(1/n)\mathbb{Z}$. The eigenvalues of L_0 on V lie in \mathbb{Z} , as does the value of $\langle h, h \rangle$. From formula (5) for the weight grading of $V(\sigma_h)$ we conclude that the weight grading and hence the conformal weight of $V(\sigma_h)$ is in $(1/n)\mathbb{Z}$. Hence σ_h is of type $n\{0\}$. One can check that $\rho(V(\sigma_h^i)) > 0$ for all $i \in \mathbb{Z}_n \setminus \{0\}$ so that V^{σ_h} satisfies the positivity assumption. Indeed, we shall show in Lemma 6.5 below that $\rho(V(\sigma_h^i)) \geq 1$ holds for each of the above cases. So we can apply the orbifold construction (see Section 2) and obtain the tame, holomorphic vertex operator algebra $V^{\text{orb}(\sigma_h)}$ of central charge 24.

The following section is aimed at the computation of a lower bound for the conformal weight of the twisted modules $V(\sigma_h^i)$, $i \in \mathbb{Z}_n \setminus \{0\}$. Since we do not know the complete vertex operator algebra structure of V at this point, the approach will be rather indirect. The idea for the combinatorial argument in the next section is from [LS16c].

6.1. Lower Bound for Conformal Weight. We continue in the above setting, considering the decomposition $V_1 = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ of V_1 into simple ideals. As explained in Section 1.3, V decomposes into a direct sum (2) of irreducible $\langle V_1 \rangle$ -modules of non-negative integer conformal weight. The conformal weight of $\langle V_1 \rangle$ is 0, and the conformal weights (given by formula (3)) of all other irreducible $\langle V_1 \rangle$ -modules appearing in the decomposition are at least 2.

Now we study the conformal weight of the unique irreducible σ_h -twisted V -module $V(\sigma_h) = V^{(h)}$. It is clear that $V(\sigma_h)$ decomposes into the direct sum of

irreducible σ_h -twisted $\langle V_1 \rangle$ -modules $M^{(h)}$ for those modules M appearing in the decomposition of V into irreducible $\langle V_1 \rangle$ -modules.

Lemma 6.1 ([LS16c], Lemma 2.7). *Let V be a tame, holomorphic vertex operator algebra of central charge 24 with $V_1 = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ semisimple and $h = (h_1, \dots, h_r)$ a semisimple element in the Cartan subalgebra \mathfrak{h} of V_1 such that σ_h has finite order. In addition we assume that*

$$(8) \quad \alpha(h_i) \geq -1$$

for all roots $\alpha \in \Phi(\mathfrak{g}_i)$ and $i = 1, \dots, r$. Let $M \cong L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, \lambda_r)$ be an irreducible $\langle V_1 \rangle$ -submodule of V . Then the conformal weight of the irreducible σ_h -twisted $\langle V_1 \rangle$ -module $M^{(h)}$ is given by

$$\rho(M^{(h)}) = \rho(M) + \sum_{i=1}^r \min\{\mu(h_i) \mid \mu \in \Pi(\lambda_i, \mathfrak{g}_i)\} + \frac{\langle h, h \rangle}{2}$$

where $\Pi(\lambda_i, \mathfrak{g}_i)$ is the set of all weights of the irreducible \mathfrak{g}_i -module $L_{\mathfrak{g}_i}(\lambda_i)$ with highest weight λ_i .

Note that condition (8) is essential for this formula. Note that since σ_h has finite order, h is a \mathbb{Q} -linear combination of the fundamental coweights.

Remark 6.2. The minimum $\min\{\mu(h) \mid \mu \in \Pi(\lambda, \mathfrak{g})\}$ for a simple Lie algebra \mathfrak{g} , a dominant integral weight $\lambda \in P_+$ and some h in the Cartan of \mathfrak{g} can be computed as follows. Clearly, the minimum does not change if we modify h by an element in the Weyl group W of \mathfrak{g} . Now assume that h is some \mathbb{R} -linear combination of the fundamental coweights. Then there is an element w in the Weyl group such that $h_- := wh$ is a \mathbb{R}_0 -linear combination of the fundamental coweights, and $\min\{\mu(h) \mid \mu \in \Pi(\lambda, \mathfrak{g})\} = \lambda(h_-)$.

It is trivial to check:

Lemma 6.3. *For each case (1) to (15) in the table on p. 19 the vector h fulfils the assumptions of Lemma 6.1. In particular, for every i , $\alpha(h_i) \geq -1$ for all roots $\alpha \in \Phi(\mathfrak{g}_i)$ of the simple component \mathfrak{g}_i of V_1 .*

The two lemmas allow us to compute the conformal weights of the possible irreducible σ_h -twisted $\langle V_1 \rangle$ -modules $M^{(h)}$ appearing in the decomposition of $V(\sigma_h) = V^{(h)}$, which gives a lower bound for $\rho(V(\sigma_h))$.

Lemma 6.4. *For each case (1) to (15) in the table on p. 19 the bound*

$$\rho(V(\sigma_h)) \geq 1$$

holds.

Proof. It suffices to show that $\rho(M^{(h)}) \geq 1$ for each irreducible component M of the decomposition (2) of V . Consider the irreducible $\langle V_1 \rangle$ -module

$$M = L_{\hat{\mathfrak{g}}_1}(k_1, \lambda_1) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, \lambda_r).$$

If $(\lambda_1, \dots, \lambda_r) = (0, \dots, 0)$, then by Lemmas 6.1 and 6.3 the conformal weight of $M^{(h)}$ is equal to $\langle h, h \rangle / 2 \geq 1$.

Now we consider $(\lambda_1, \dots, \lambda_r) \neq (0, \dots, 0)$, in which case $\rho(M) \in \mathbb{Z}_{\geq 2}$. Using Lemmas 6.1 and 6.3 it is straightforward to compute the conformal weight $\rho(M^{(h)})$ for every irreducible $\langle V_1 \rangle$ -module M . In the all cases except (11) and (15) we find that $\rho(M^{(h)}) \geq 1$ for those irreducible modules M satisfying $\rho(M) \in \mathbb{Z}_{\geq 2}$.

Cases (11) and (15) require more involved arguments. We begin with case (11) where $V_1 = A_{4,5}^2$ and $h = (0, (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee)/5)$. Here, there are eleven pairs

(λ_1, λ_2) that cannot be excluded by the simple argument above. More precisely, if the irreducible $\langle V_1 \rangle$ -module M has highest weight among

$$([0, 0, 0, 0], [0, 2, 2, 0]), ([0, 0, 0, 0], [1, 0, 2, 2]), ([0, 0, 0, 0], [2, 2, 0, 1]),$$

then $\rho(M^{(h)}) = 3/5$, and if M has highest weight among

$$\begin{aligned} &([0, 0, 0, 1], [0, 1, 2, 1]), ([0, 0, 0, 1], [1, 2, 1, 0]), \\ &([0, 0, 0, 1], [2, 0, 1, 2]), ([0, 0, 0, 1], [2, 1, 0, 2]), \\ &([1, 0, 0, 0], [0, 1, 2, 1]), ([1, 0, 0, 0], [1, 2, 1, 0]), \\ &([1, 0, 0, 0], [2, 0, 1, 2]), ([1, 0, 0, 0], [2, 1, 0, 2]), \end{aligned}$$

then $\rho(M^{(h)}) = 4/5$. Here we have written the weight $\sum_{i=1}^l m_i \Lambda_i$ as $[m_1, \dots, m_l]$ in terms of the fundamental weights. In all these cases $\rho(M) = 2$.

The vertex operator algebra $\langle V_1 \rangle \cong (L_{A_4}(5, 0))^{\otimes 2}$ has 146 irreducible modules M of conformal weight $\rho(M) = 2$. The identities listed in [EMS15], Theorem 6.1, are equivalent to a linear system of equations on the subset of 146 corresponding multiplicities $m_{(\lambda_1, \lambda_2)}$ in the decomposition (2). We seek solutions in non-negative integers to this system. An application of the simplex algorithm shows that all multiplicities vanish except possibly those of the weights

$$\begin{aligned} &([0, 0, 0, 0], [0, 0, 0, 5]), ([0, 1, 1, 0], [1, 2, 0, 0]), \\ &([2, 0, 1, 0], [2, 0, 1, 0]), ([0, 1, 1, 0], [2, 0, 0, 2]) \end{aligned}$$

and of their images under the action of the group \mathbb{Z}_2^2 generated by exchange of the two copies of A_4 and the involution $\Lambda_i \mapsto \Lambda_{4-i}$, $i = 1, \dots, 4$. Hence, none of the above eleven problematic modules can occur.

In case (15) with $V_1 = A_{1,2}A_{3,4}^3$ and $h = (\frac{\Lambda_1^\vee}{2}, \frac{\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_3^\vee}{8}, \frac{\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_3^\vee}{8}, 0)$ we proceed similarly. Here, there are seventeen tuples $(\lambda_1, \dots, \lambda_4)$ that cannot be excluded by the simple combinatorial argument, namely

$$\begin{aligned} &([0], [2, 1, 0], [2, 1, 0], [0, 0, 0]), \\ &([1], [1, 0, 1], [3, 0, 1], [0, 0, 0]), ([1], [2, 0, 0], [2, 0, 2], [0, 0, 0]), \\ &([1], [2, 0, 2], [2, 0, 0], [0, 0, 0]), ([1], [3, 0, 1], [1, 0, 1], [0, 0, 0]), \\ &([2], [1, 0, 1], [2, 1, 0], [0, 0, 0]), ([2], [1, 1, 1], [2, 0, 0], [0, 0, 0]), \\ &([2], [2, 0, 0], [1, 1, 1], [0, 0, 0]), ([2], [2, 1, 0], [1, 0, 1], [0, 0, 0]) \end{aligned}$$

with $\rho(M^{(h)}) = 3/4$ and

$$\begin{aligned} &([0], [1, 0, 1], [4, 0, 0], [0, 0, 0]), ([0], [4, 0, 0], [1, 0, 1], [0, 0, 0]), \\ &([1], [0, 1, 0], [4, 0, 0], [0, 0, 0]), ([1], [3, 0, 1], [4, 0, 0], [0, 0, 0]), \\ &([1], [4, 0, 0], [0, 1, 0], [0, 0, 0]), ([1], [4, 0, 0], [3, 0, 1], [0, 0, 0]), \\ &([2], [2, 1, 0], [4, 0, 0], [0, 0, 0]), ([2], [4, 0, 0], [2, 1, 0], [0, 0, 0]) \end{aligned}$$

with $\rho(M^{(h)}) = 7/8$. In all these cases $\rho(M) \in \{2, 3\}$.

Again we show that none of these modules occur in the decomposition of V into irreducible modules for $\langle V_1 \rangle \cong L_{A_1}(2, 0) \otimes (L_{A_3}(4, 0))^{\otimes 3}$. Indeed, as described in the proof of Theorem 1.1, the modular invariance implies a large system of linear equations on the multiplicities $m_{(\lambda_1, \dots, \lambda_4)}$ occurring in the decomposition (2). In

the present case the system is underdetermined, but somewhat surprisingly the sum $m_{\lambda^{(1)}} + m_{\lambda^{(2)}} + m_{\lambda^{(3)}}$ of the multiplicities of the modules with highest weights

$$\begin{aligned}\lambda^{(1)} &= ([0], [0, 0, 0], [0, 0, 0], [0, 4, 0]), \\ \lambda^{(2)} &= ([0], [0, 0, 0], [0, 4, 0], [0, 0, 0]), \\ \lambda^{(3)} &= ([0], [0, 4, 0], [0, 0, 0], [0, 0, 0])\end{aligned}$$

is forced to be 3.

We observe that each of these modules is a simple current of order 2. Recall that a *simple current* is an irreducible module M with the property that fusion with M acts by permutation on the set of all irreducible modules.

In general, if a simple current M occurs in the decomposition (2), then the multiplicities are constant along orbits under fusion by M . In particular, the multiplicity of M itself equals that of $\langle V_1 \rangle \cong L_{\hat{\mathfrak{g}}_1}(k_1, 0) \otimes \dots \otimes L_{\hat{\mathfrak{g}}_r}(k_r, 0)$, which is 1. It follows in the present case that $m_{\lambda^{(1)}} = m_{\lambda^{(2)}} = m_{\lambda^{(3)}} = 1$. There is therefore an action of \mathbb{Z}_2^3 on the set of irreducible $\langle V_1 \rangle$ -modules by permutations, under which the multiplicities of (2) are constant.

We treat these equalities as extra equations to be added to the system. In the reduced system we observe that the sum $m_{\lambda^{(4)}} + m_{\lambda^{(5)}} + m_{\lambda^{(6)}}$ of the multiplicities of the modules with highest weights

$$\begin{aligned}\lambda^{(4)} &= ([2], [0, 0, 0], [0, 0, 0], [0, 0, 4]), \\ \lambda^{(5)} &= ([2], [0, 0, 0], [0, 0, 4], [0, 0, 0]), \\ \lambda^{(6)} &= ([2], [0, 0, 4], [0, 0, 0], [0, 0, 0])\end{aligned}$$

is forced to be 3. Again, these modules are simple currents, so we conclude that $m_{\lambda^{(4)}} = m_{\lambda^{(5)}} = m_{\lambda^{(6)}} = 1$, and we supplement the system with equations coming from the action by fusion of these modules.

The system, thus supplemented, has a unique solution in positive integers. The corresponding decomposition (2) is exactly the one given in [Sch93].

We can easily verify that every irreducible $\langle V_1 \rangle$ -module M occurring in (2) with positive multiplicity satisfies the property $\rho(M^{(h)}) \geq 1$, i.e. none of the above seventeen problematic modules occurs. \square

We would like to extend the result of Lemma 6.4 to the twisted V -modules $V(\sigma_h^i)$ for all $i \in \mathbb{Z}_n \setminus \{0\}$. For $n = 2$ we are done, and for $n = 3$ it suffices to recall that the contragredient module $(V(\sigma_h))' \cong V(\sigma_h^{-1}) = V(\sigma_h^2)$ has the same grading as $V(\sigma_h)$.

For $n > 3$ we have to consider modules twisted by $\sigma_h^i = \sigma_{ih}$ for $i > 1$. In general, condition (8) is not satisfied by $ih = (ih_1, \dots, ih_r)$, so Lemma 6.1 cannot be applied directly. To get around this difficulty we observe that V is graded by the weight lattice relative to the action of \mathfrak{h} . Hence for any element k of the coroot lattice Q^\vee , the automorphisms σ_h and σ_{h+k} of V coincide. Furthermore it is always possible to find a representative $i[h]$ of $ih + Q^\vee$ satisfying the condition (8), i.e.

$$\alpha(i[h]_j) \geq -1$$

for all roots $\alpha \in \Phi(\mathfrak{g}_j)$ and all $j = 1, \dots, r$. (The element $i[h]$ is in general not unique.)

Lemma 6.5. *For the cases (1) to (15) in the table on p. 19*

$$\rho(V(\sigma_h^i)) \geq 1$$

for all $i \in \mathbb{Z}_n \setminus \{0\}$.

Proof. We proceed as described above, choosing the following representatives $i[h]$:

(11) For $V_1 \cong A_{4,5}^2$

$$\begin{aligned} h &= (0, \frac{\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee}{5}), \\ 2[h] &= (0, \frac{-3\Lambda_1^\vee + 2\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee}{5}), \\ 3[h] &= (0, \frac{3\Lambda_1^\vee - 2\Lambda_2^\vee - 2\Lambda_3^\vee + 3\Lambda_4^\vee}{5}), \\ 4[h] &= (0, \frac{-\Lambda_1^\vee - \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee}{5}). \end{aligned}$$

(12) For $V_1 \cong A_4 A_{9,2} B_3$

$$\begin{aligned} h &= (0, \frac{\Lambda_1^\vee + 3\Lambda_3^\vee}{4}, 0), \\ 2[h] &= (0, \frac{-\Lambda_1^\vee - \Lambda_3^\vee}{2} + \Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee - \Lambda_7^\vee + \Lambda_8^\vee - \Lambda_9^\vee, 0), \\ 3[h] &= (0, \frac{3\Lambda_1^\vee - 3\Lambda_3^\vee}{4} + \Lambda_7^\vee - \Lambda_8^\vee, 0). \end{aligned}$$

(13) For $V_1 \cong B_6 C_{10}$

$$\begin{aligned} h &= (0, \frac{\Lambda_2^\vee}{4} + \frac{\Lambda_{10}^\vee}{2}), \\ 2[h] &= (0, \frac{\Lambda_2^\vee}{2} - \Lambda_3^\vee + \Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee - \Lambda_7^\vee + \Lambda_8^\vee - \Lambda_9^\vee + \Lambda_{10}^\vee), \\ 3[h] &= (0, \frac{3\Lambda_2^\vee}{4} - \frac{\Lambda_{10}^\vee}{2}). \end{aligned}$$

(14) For $V_1 \cong A_1 C_{5,3} G_{2,2}$

$$\begin{aligned} h &= (0, \frac{\Lambda_4^\vee}{6} + \frac{2\Lambda_5^\vee}{3}, \frac{\Lambda_1^\vee}{3}), \\ 2[h] &= (0, \frac{\Lambda_4^\vee}{3} - \frac{2\Lambda_5^\vee}{3}, \frac{2\Lambda_1^\vee}{3} - \Lambda_2^\vee), \\ 3[h] &= (0, -\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee + \frac{\Lambda_4^\vee}{2}, 0), \\ 4[h] &= (0, -\frac{\Lambda_4^\vee}{3} + \frac{2\Lambda_5^\vee}{3}, -\frac{2\Lambda_1^\vee}{3} + \Lambda_2^\vee), \\ 5[h] &= (0, -\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee + \frac{5\Lambda_4^\vee}{6} - \frac{2\Lambda_5^\vee}{3}, \frac{2\Lambda_1^\vee}{3} - \Lambda_2^\vee). \end{aligned}$$

(15) For $V_1 \cong A_{1,2}A_{3,4}^3$

$$\begin{aligned}
h &= \left(\frac{\Lambda_1^\vee}{2}, \frac{\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_3^\vee}{8}, \frac{\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_3^\vee}{8}, 0 \right), \\
2[h] &= \left(\Lambda_1^\vee, \frac{-3\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee}{4}, \frac{-3\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee}{4}, 0 \right), \\
3[h] &= \left(-\frac{\Lambda_1^\vee}{2}, \frac{-5\Lambda_1^\vee + 3\Lambda_2^\vee + \Lambda_3^\vee}{8}, \frac{-5\Lambda_1^\vee + 3\Lambda_2^\vee + \Lambda_3^\vee}{8}, 0 \right), \\
4[h] &= \left(0, \frac{\Lambda_1^\vee - \Lambda_2^\vee - \Lambda_3^\vee}{2}, \frac{\Lambda_1^\vee - \Lambda_2^\vee - \Lambda_3^\vee}{2}, 0 \right), \\
5[h] &= \left(\frac{\Lambda_1^\vee}{2}, \frac{5\Lambda_1^\vee - 3\Lambda_2^\vee - \Lambda_3^\vee}{8}, \frac{5\Lambda_1^\vee - 3\Lambda_2^\vee - \Lambda_3^\vee}{8}, 0 \right), \\
6[h] &= \left(-\Lambda_1^\vee, \frac{3\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee}{4}, \frac{3\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee}{4}, 0 \right), \\
7[h] &= \left(-\frac{\Lambda_1^\vee}{2}, \frac{-\Lambda_1^\vee - \Lambda_2^\vee - 3\Lambda_3^\vee}{8}, \frac{-\Lambda_1^\vee - \Lambda_2^\vee - 3\Lambda_3^\vee}{8}, 0 \right).
\end{aligned}$$

Then we repeat the proof of Lemma 6.4 to show that $\rho(V(\sigma_{i[h]})) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$. For all cases except for (11) and (15) this works without complication. In cases (11) and (15) we make use of the same information on the decomposition of V as a $\langle V_1 \rangle$ -module that we used in the proof of Lemma 6.4. \square

6.2. Dimension Formula. With Lemma 6.5 in hand we may apply the dimension formula Corollary 3.3 to compute $d = \dim(V_1^{\text{orb}(\sigma_h)})$ in our fifteen cases. This information, along with Kac's theory (see Section 1.7), will permit us to determine the Lie algebra structure of $V_1^{\text{orb}(\sigma_h)}$.

Proposition 6.6 (Part 2). *Let V and σ_h as in the table on p. 19. Then the Lie algebra structure of $V_1^{\text{orb}(\sigma_h)}$ is as follows:*

No.	V_1	n	$V_1^{\sigma_h}$	d	$V_1^{\text{orb}(\sigma_h)}$
(1)	$A_5C_5E_{6,2}$	2	$A_5C_5D_5\mathbb{C}$	264	$A_9^2D_6$
(2)	$A_3A_{7,2}C_3^2$	2	$A_1^2A_3A_7B_2^2$	216	$A_7^2D_5^2$
(3)	$A_{8,2}F_{4,2}$	2	A_8B_4	240	A_8^3
(4)	$B_8E_{8,2}$	2	D_8E_8	744	E_8^3
(5)	$A_2^2A_{5,2}^2B_2$	2	$A_1A_2^2A_3A_5B_2\mathbb{C}$	168	$A_5^4D_4$
(6)	$C_8F_4^2$	2	$C_4^2F_4^2$	312	E_6^4
(7)	$A_{4,2}^2C_{4,2}$	2	$A_4^2B_2^2$	144	A_4^6
(8)	$A_{2,2}^4D_{4,4}$	2	$A_1^4A_2^4$	96	A_2^{12}
(9)	$B_5E_{7,2}F_4$	2	$A_1B_5D_6F_4$	312	$A_{11}D_7E_6$
(10)	$B_4C_6^2$	2	$B_2B_4C_4C_6$	312	$A_{11}D_7E_6$
(11)	$A_{4,5}^2$	5	$A_4\mathbb{C}^4$	144	A_4^6
(12)	$A_4A_{9,2}B_3$	4	$A_1A_4A_7B_3\mathbb{C}$	456	$D_{10}E_7^2$
(13)	B_6C_{10}	4	$A_7B_2B_6\mathbb{C}$	456	$D_{10}E_7^2$
(14)	$A_1C_{5,3}G_{2,2}$	6	$A_1A_2C_4\mathbb{C}$	312	E_6^4
(15)	$A_{1,2}A_{3,4}^3$	8	$A_3\mathbb{C}^7$	168	D_4^6

Moreover, $V^{\text{orb}(\sigma_h)}$ is isomorphic as vertex operator algebra to V_L where L is exactly the lattice in Proposition 5.1.

Proof. The dimensions of the fixed-point Lie subalgebras $V_1^{\sigma_h^k}$ are easily computable, and so we may use the dimension formulae (Corollary 3.3) to compute

$d = \dim(V_1^{\text{orb}(\sigma_h)})$. This yields the values for d listed in the table above. We observe that in each case $\dim(V_1^{\text{orb}(\sigma_h)}) = \dim((V_L)_1)$.

The Lie algebra $V_1^{\sigma_h}$ has to be a fixed-point Lie subalgebra of $V_1^{\text{orb}(\sigma_h)}$ under an automorphism of order dividing n , namely under the inverse orbifold automorphism with respect to σ_h restricted to $V_1^{\text{orb}(\sigma_h)}$. Then, using Kac's theory we determine that the Lie algebra listed in the proposition for $V_1^{\text{orb}(\sigma_h)}$ is the only Lie algebra of dimension d in Schellekens' list of which $V_1^{\sigma_h}$ is a possible fixed-point Lie subalgebra under an automorphism of order dividing n . This determines $V_1^{\text{orb}(\sigma_h)}$.

In all of the cases, the Lie algebra $V_1^{\text{orb}(\sigma_h)}$ has rank 24. By Corollary 1.4 in [DM04b] we conclude that $V^{\text{orb}(\sigma_h)}$ is isomorphic to a lattice vertex operator algebra V_L . Indeed, we see that L is exactly the Niemeier lattice in the corresponding entry of the table in Proposition 5.1. \square

Remark 6.7. For the cases of order n a prime, i.e. $n = 2, 5$, there is a nice alternative argument by symmetry. Recall that $V_1 = (V_L^{\text{orb}(\hat{\nu})})_1$ by definition of V and $V_1^{\sigma_h} \cong (V_L^{\hat{\nu}})_1$ by the choice of σ_h . In particular the dimensions coincide. Corollary 3.3 applied to the orbifold construction with V and $\sigma_h \in \text{Aut}(V)$ and to the one with V_L and $\hat{\nu}$ yields

$$\begin{aligned} \dim(V_1) + \dim(V_1^{\text{orb}(\sigma_h)}) &= 24 + (n+1) \dim(V_1^{\sigma_h}), \\ \dim((V_L)_1) + \dim((V_L^{\text{orb}(\hat{\nu})})_1) &= 24 + (n+1) \dim((V_L^{\hat{\nu}})_1) \end{aligned}$$

so that with the above observation we obtain

$$\dim((V_L)_1) = \dim(V_1^{\text{orb}(\sigma_h)}).$$

7. PART 3: UNIQUE CONJUGACY CLASS

In this section we establish the uniqueness up to conjugacy of automorphisms of lattice vertex operator algebras satisfying certain conditions.

7.1. Lemmas. We begin with some general lemmas. The following is a generalisation of Lemma 3.6 in [LS16c]:

Lemma 7.1. *Let \mathfrak{g} be a Lie algebra with a direct-sum decomposition*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$$

for some $n \in \mathbb{Z}_{>0}$, and assume that g is an automorphism of \mathfrak{g} of order n such that

$$g(\mathfrak{g}_1) = \mathfrak{g}_2, \dots, g(\mathfrak{g}_{n-1}) = \mathfrak{g}_n, g(\mathfrak{g}_n) = \mathfrak{g}_1$$

(so that in particular, all the ideals \mathfrak{g}_i , $i = 1, \dots, n$, are isomorphic). Let $f \in \text{Inn}(\mathfrak{g})$ be an inner automorphism of \mathfrak{g} , and assume that $(fg)^n = 1$. Then fg is conjugate to g under an inner automorphism of \mathfrak{g} .

Proof. We can write f uniquely as $f = f_1 f_2 \dots f_n$ where $f_i \in \text{Inn}(\mathfrak{g}_i)$ is an inner automorphism of \mathfrak{g}_i and the f_i commute pairwise. Let us define $a^b := b^{-1}ab$. Then $f_i^{g^j} = g^{-j} f_i g^j \in \text{Inn}(\mathfrak{g}_{i-j})$ where we view the subtraction $i-j$ modulo n .

Using $g^n = 1$ we rewrite the relation $(fg)^n = 1$ as

$$\begin{aligned} 1 &= (fg)^n = (f_1 f_2 f_3 g)^n \\ &= (f_1 f_2 \dots f_n) (f_1^{g^{n-1}} f_2^{g^{n-1}} \dots f_n^{g^{n-1}}) \dots (f_1^g f_2^g \dots f_n^g), \end{aligned}$$

which we reorder using the commutativity to obtain

$$1 = \underbrace{(f_1 f_n^{g^{n-1}} f_{n-1}^{g^{n-2}} \dots f_2^g)}_{\in \text{Inn}(\mathfrak{g}_1)} \underbrace{(f_2 f_1^{g^{n-1}} f_n^{g^{n-2}} \dots f_3^g)}_{\in \text{Inn}(\mathfrak{g}_2)} \dots \underbrace{(f_n f_{n-1}^{g^{n-1}} \dots f_1^g)}_{\in \text{Inn}(\mathfrak{g}_n)}.$$

This implies that

$$\begin{aligned} f_1 f_n^{g^{n-1}} f_{n-1}^{g^{n-2}} \cdots f_2^g &= 1, \\ f_2 f_1^{g^{n-1}} f_n^{g^{n-2}} \cdots f_3^g &= 1, \\ &\vdots \\ f_n f_{n-1}^{g^{n-1}} \cdots f_1^g &= 1. \end{aligned}$$

We set

$$x := (f_1)(f_2 f_1^{g^{n-1}})(f_3 f_2^{g^{n-1}} f_1^{g^{n-2}}) \cdots (f_{n-1} f_{n-2}^{g^{n-1}} \cdots f_1^{g^2}),$$

where the i -th bracket is in $\text{Inn}(\mathfrak{g}_i)$ and hence all brackets commute. Using the above relations the inverse of x is given by

$$x^{-1} = (f_n^{g^{n-1}} f_{n-1}^{g^{n-2}} \cdots f_2^g) \cdots (f_n^{g^2} f_{n-1}^g)(f_n^g).$$

A simple calculation shows that $xgx^{-1} = fg$. \square

Recall that $\langle \cdot, \cdot \rangle$ also denotes the symmetric bilinear form on a lattice L or on its complexification $L_{\mathbb{C}} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and that we identify $L_{\mathbb{C}}$ with $\{h(-1) \otimes \mathfrak{e}_0 \mid h \in L_{\mathbb{C}}\} \subseteq (V_L)_1$.

Lemma 7.2 (cf. Lemma 3.9 in [LS16c]). *Let L be a positive-definite, even lattice and R the sublattice of L generated by the vectors $\alpha \in L$ of norm $\langle \alpha, \alpha \rangle / 2 = 1$, which we suppose to be of full rank. Let g be an automorphism of the lattice vertex operator algebra V_L whose restriction to the reductive Lie algebra $(V_L)_1$ is trivial. Then $g = e^{(2\pi i)h_0}$ for some $h \in R'$, the dual lattice of R .*

Proof. In particular, g acts trivially on the Cartan subalgebra $\mathfrak{h} = \{h(-1) \otimes \mathfrak{e}_0 \mid h \in L_{\mathbb{C}}\}$ of $(V_L)_1$. By Lemma 2.5 in [DN99] g is of the form $g = e^{(2\pi i)h_0}$ for some $h \in L_{\mathbb{C}}$. But h acts as multiplication by $\langle h, \alpha \rangle$ on the graded piece $V_L(\alpha) \subseteq V_L$ and, by assumption, $e^{(2\pi i)\langle h, \alpha \rangle} = 1$ for all $\alpha \in L$ of norm $\langle \alpha, \alpha \rangle / 2 = 1$. This means that $\langle h, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in L$ of norm $\langle \alpha, \alpha \rangle / 2 = 1$ and hence by linearity for all $\alpha \in R$. This proves the statement. \square

The following Lemma generalises Lemma 3.7 in [LS16c].

Lemma 7.3. *Let V be a vertex operator algebra of CFT-type and assume that the Lie algebra V_1 is reductive. Let \mathfrak{h} be a Cartan subalgebra of V_1 and g an automorphism of V of order $n \in \mathbb{Z}_{>0}$ such that $g(\mathfrak{h}) = \mathfrak{h}$. For $u \in \mathfrak{h} \subseteq V_1$ consider the inner automorphism $e^{u_0} \in \text{Aut}(V)$. Then $e^{u_0}g$ is conjugate in $\text{Aut}(V)$ to $\exp((\frac{1}{n} \sum_{i=0}^{n-1} g^i u)_0)g$ under an inner automorphism of V .*

We observe that $u \mapsto \frac{1}{n} \sum_{i=0}^{n-1} g^i u$ is a projection from \mathfrak{h} to the subspace of fixed points under g .

Proof. Note that all elements of \mathfrak{h} commute with each other so that $[u_0, v_0] = 0$ for $u, v \in \mathfrak{h}$ by Borcherds' identity. Moreover, $ge^{u_0} = e^{(gu)_0}g$ and $(e^{u_0})^{-1} = e^{(-u)_0}$ for any $u \in \mathfrak{h}$. We define $x := \exp((\frac{1}{n} \sum_{i=1}^{n-1} ig^i u)_0)$ and calculate

$$\begin{aligned} xe^{u_0}gx^{-1} &= \exp((\frac{1}{n} \sum_{i=1}^{n-1} ig^i u)_0) \exp(u_0)g \exp((-\frac{1}{n} \sum_{i=1}^{n-1} ig^i u)_0) \\ &= \exp((\frac{1}{n} \sum_{i=1}^n ig^i u)_0) \exp((-\frac{1}{n} \sum_{i=1}^{n-1} ig^{i+1} u)_0)g = \exp((\frac{1}{n} \sum_{i=0}^{n-1} g^i u)_0)g. \end{aligned}$$

This proves the statement. \square

In the special case of a lattice vertex operator algebra V_L associated with a positive-definite, even lattice L choose the Cartan $\mathfrak{h} = \{h(-1) \otimes \mathfrak{e}_0 \mid h \in L_{\mathbb{C}}\} \cong L_{\mathbb{C}}$ and let $\hat{\nu} \in \text{Aut}(V_L)$ be an automorphism obtained as lift of a lattice automorphism $\nu \in \text{Aut}(L)$. The above lemma states that $e^{h_0}\hat{\nu}$ is conjugate to $e^{(\pi_{\nu}h)_0}\hat{\nu}$ where π_{ν} is the projection from $L_{\mathbb{C}}$ to the complexified fixed-point sublattice $L^{\nu} \otimes \mathbb{C}$.

Lemma 7.4. *Let \mathfrak{g} be a simple Lie algebra not of type D_4 . Then any two outer automorphisms of \mathfrak{g} conjugate in $\text{Aut}(\mathfrak{g})$ are in fact conjugate under an element of $\text{Inn}(\mathfrak{g})$ (and also under an element of $\text{Aut}(\mathfrak{g}) \setminus \text{Inn}(\mathfrak{g})$ if the latter is non-empty).*

Proof. If \mathfrak{g} is not of type D_4 , then $\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$ is either trivial or isomorphic to \mathbb{Z}_2 . In the trivial case there is nothing to show, so assume the latter. In general, let G be a group and $H \subseteq G$ a subgroup of index 2. Let $a, b \in G \setminus H$ be conjugate under $x \in G$, i.e. $a = xbx^{-1}$. Then $a = (xb)b(xb)^{-1}$ too, and one of x and xb lies in H while the other lies in $G \setminus H$. \square

7.2. Uniqueness. In this section we use the above lemmas to prove the main result of this section:

Proposition 7.5 (Part 3). *Let $L = N(Q)$ be a Niemeier lattice with root lattice Q , V_L the associated lattice vertex operator algebra and g an automorphism of V_L . Then for each entry of the following table the order n of g , the weight-one fixed-point Lie subalgebra $(V_L^g)_1$ and, if indicated, the value of $\rho(V_L(g))$ determine a unique conjugacy class in $\text{Aut}(V_L)$:*

No.	L	n	$(V_L^g)_1$	$\rho(V_L(g))$
(1)	$N(A_9^2 D_6)$	2	$A_5 C_5 D_5 \mathbb{C}$	1
(2)	$N(A_7^2 D_5^2)$	2	$A_1^2 A_3 A_7 B_2^2$	
(3)	$N(A_8^3)$	2	$A_8 B_4$	
(4)	$N(E_8^3)$	2	$D_8 E_8$	
(5)	$N(A_5^4 D_4)$	2	$A_1 A_2^2 A_3 A_5 B_2 \mathbb{C}$	
(6)	$N(E_6^4)$	2	$C_4^2 F_4^2$	
(7)	$N(A_4^6)$	2	$A_4^2 B_2^2$	
(8)	$N(A_2^{12})$	2	$A_1^4 A_2^4$	
(9)	$N(A_{11} D_7 E_6)$	2	$A_1 B_5 D_6 F_4$	
(10)	$N(A_{11} D_7 E_6)$	2	$B_2 B_4 C_4 C_6$	1
(11)	$N(A_4^6)$	5	$A_4 \mathbb{C}^4$	1
(12)	$N(D_{10} E_7^2)$	4	$A_1 A_4 A_7 B_3 \mathbb{C}$	
(13)	$N(D_{10} E_7^2)$	4	$A_7 B_2 B_6 \mathbb{C}$	
(14)	$N(E_6^4)$	6	$A_1 A_2 C_4 \mathbb{C}$	
(15)	$N(D_4^6)$	8	$A_3 \mathbb{C}^7$	1

In particular, the automorphism $\hat{\nu} \in \text{Aut}(V_L)$ from Proposition 5.1 and the inverse orbifold automorphism $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_h)})$ with respect to the inner automorphism σ_h (see table on p. 19) are conjugate under the isomorphism $V^{\text{orb}(\sigma_h)} \cong V_L$.

The proof of the proposition will be carried out in several steps. The first step is to use Kac's theory on fixed-point Lie subalgebras of simple Lie algebras (see Section 1.7) to establish an analogous result at the level of conjugacy in $\text{Aut}((V_L)_1)$.

Lemma 7.6. *For each entry of the table in Proposition 7.5 there is a unique conjugacy class in $\text{Aut}((V_L)_1)$ of order n with fixed-point Lie subalgebra $(V_L^g)_1$.*

Proof. We apply the straightforward extension of Kac's theory to semisimple Lie algebras (see Section 1.7) case by case. \square

In the following we investigate whether two automorphisms of $(V_L)_1$ of order n with the same fixed-point Lie subalgebra are not just conjugate in $\text{Aut}((V_L)_1)$ but more specifically conjugate under an element whose projection to $\text{Out}((V_L)_1) = \text{Aut}((V_L)_1)/\text{Inn}((V_L)_1)$ lies in a certain subgroup $H \subseteq \text{Out}((V_L)_1)$ that we shall describe in the following (see Lemma 7.8). This is in particular fulfilled if the two automorphisms are conjugate under an inner automorphism, which corresponds to the trivial element in $\text{Out}((V_L)_1)$.

The following summary is based on Sections 4.3, 16.1 and 18.4 of [CS99] and partially draws from [ISS15], Section 2 and [LS16c], Section 3.3. Let $L = N(Q)$ be a Niemeier lattice other than the Leech lattice with root lattice Q . We denote by H the subgroup of $\text{Aut}(L)$ consisting of all elements that preserve the simple roots of the root system of Q as a set. The elements of H can be naturally viewed as automorphisms of the Dynkin diagram of Q , but there are in general Dynkin diagram automorphisms that do not arise from an element in $\text{Aut}(L)$. By definition, the group H is isomorphic to the automorphism group of the finite quadratic space L/Q , the glue code of L . Recall that the outer automorphism group $\text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g})$ of a finite-dimensional, semisimple Lie algebra \mathfrak{g} is naturally isomorphic to the automorphism group of the Dynkin diagram of \mathfrak{g} . Hence, H can be naturally identified with a subgroup of $\text{Out}((V_L)_1)$.

Moreover, one can show that $H \cong \text{Aut}(L)/W$ where $W = G_0$ is the Weyl group of L (or that of Q), i.e. the subgroup of $\text{Aut}(L)$ generated by the reflections in the roots of L . $\text{Aut}(L)$ is a split group extension $W:H$.

Let G_1 be the normal subgroup of H consisting of those automorphisms that fix each irreducible component of the Dynkin diagram of Q . Then H is a group extension $G_1.G_2$ where $G_2 = H/G_1$ can be naturally realised as a subgroup of the group of permutations of the isomorphic components of the Dynkin diagram of Q .

The Weyl group $G_0 = W$ and the groups G_1 and G_2 are tabulated in [CS99] for all Niemeier lattices.

Now let V_L be the lattice vertex operator algebra associated with $L = N(Q)$. Recall that $K = \langle \{e^{v_0} \mid v \in (V_L)_1\} \rangle$ denotes the group of inner automorphisms of V_L . There is a natural homomorphism

$$p : \text{Aut}(V_L) \rightarrow H$$

and its kernel is K , i.e. $\text{Aut}(V_L)/K \cong H$ (see [LS16c], Lemma 3.10). Let

$$r : \text{Aut}(V_L) \rightarrow \text{Aut}((V_L)_1)$$

denote the restriction from V_L to $(V_L)_1$. Moreover, consider for any Lie algebra \mathfrak{g} the natural projection

$$\pi : \text{Aut}(\mathfrak{g}) \rightarrow \text{Out}(\mathfrak{g}) = \text{Aut}(\mathfrak{g})/\text{Inn}(\mathfrak{g}).$$

Finally, we denote by q the composition

$$q : \text{Aut}(V_L) \rightarrow H \rightarrow G_2.$$

For an automorphism g of V_L , $p(g) \in H$ can be identified with the element $\pi(r(g)) \in \text{Out}((V_L)_1)$, i.e. there is the following commutative diagram:

$$(9) \quad \begin{array}{ccc} \text{Aut}(V_L) & \xrightarrow{p} & H \\ \downarrow r & & \downarrow \\ \text{Aut}((V_L)_1) & \xrightarrow{\pi} & \text{Out}((V_L)_1) \end{array}$$

The restriction $r : \text{Aut}(V_L) \rightarrow \text{Aut}((V_L)_1)$ is in general not surjective. Since p is surjective, $r(\text{Aut}(V_L)) = \pi^{-1}(H)$, i.e. the group $\pi^{-1}(H)$ can be identified with the subgroup of $\text{Aut}((V_L)_1)$ of automorphisms that lift to automorphisms of V_L .

Lemma 7.7. *Under the assumptions of Proposition 7.5 the projection $p(g)$ is in a unique conjugacy class in H .*

Proof. We make a case-by-case analysis. In all of the cases, since the rank of $(V_L^g)_1$ is less than the rank of $(V_L)_1$, the automorphism $r(g)$ of the Lie algebra $(V_L)_1$ is outer, so $p(g)$ is non-trivial. If n is prime, i.e. $n = 2, 5$, we deduce that $p(g)$ has order $m = n$. For the two cases in which g has order $n = 4$ we use Kac's theory to determine that $p(g)$ has order $m = 2$, in the order 6 case $m = n = 6$, and in the order $n = 8$ case $p(g)$ has order $m = 4$.

In each case we determine the group structure of H . If H possesses a unique conjugacy class of elements of order m , then we are done. If not, we use Kac's theory to determine all conjugacy classes of Lie algebra automorphisms of $(V_L)_1$ with fixed-point Lie subalgebra $(V_L^g)_1$. There can be more than one such conjugacy class in $\text{Aut}((V_L)_1)$, but in all of the following cases they reduce to a single conjugacy class in H .

Let τ denote the diagram automorphism of order 2 of A_l for $l \geq 2$, D_l for $l \geq 5$ and E_6 . For a generic simple type X_l and $\{p_1, \dots, p_s\} = \{1, 2, \dots, s\}$ we denote by $\sigma_{(p_1 p_2 \dots p_s)}$ the permutation of the simple components of X_l^s of cycle structure $(p_1 p_2 \dots p_s)$.

- (1) For $L = N(A_9^2 D_6)$ the group $H \cong \mathbb{Z}_4$ has exactly one element of order 2.
- (2) For $L = N(A_7^2 D_5^2)$ the group $H \cong D_4$ (the dihedral group of order 8) has exactly three conjugacy classes of order 2. These can be distinguished by their projection to G_2 :

conj. class in H		proj. to G_2	
order	length	on A_7^2	on D_5^2
2	1	id	id
2	2	id	$\sigma_{(12)}$
2	2	$\sigma_{(12)}$	id

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong A_7^2 D_5^2$ of order 2 with fixed-point Lie subalgebra $A_1^2 A_3 A_7 B_2^2$ can only occur as

$$A_7^2 \rightsquigarrow A_7, \quad D_5 \rightsquigarrow A_1^2 A_3, \quad D_5 \rightsquigarrow B_2^2$$

up to reordering of the two copies of D_5 with

$$q(g) = \begin{cases} \sigma_{(12)} & \text{on } A_7^2, \\ \text{id} & \text{on } D_5^2 \end{cases}$$

so that $p(g)$ has to be in the third conjugacy class.

- (3) For $L = N(A_8^3)$ the group $H \cong D_6$ (the dihedral group of order 12) has exactly three conjugacy classes of order 2. In this case it is not enough to study the projection to G_2 . The conjugacy classes can be described as:

conj. class in H		on A_8^3
order	length	
2	1	$\tau_{(1)} \tau_{(2)} \tau_{(3)}$
2	3	$\sigma_{(12)}, \sigma_{(23)}, \sigma_{(13)}$
2	3	$\sigma_{(12)} \tau_{(3)}, \sigma_{(23)} \tau_{(1)}, \sigma_{(13)} \tau_{(2)}$

where $\tau_{(i)}$ denotes the diagram automorphism τ (see above) on the i -th copy $A_8^{(i)}$ of A_8 , $i = 1, 2, 3$.

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong A_8^3$ of order 2 with fixed-point Lie subalgebra $A_8 B_4$ can only occur as

$$A_8^2 \rightsquigarrow A_8, \quad A_8 \rightsquigarrow B_4$$

up to reordering of the three copies of A_8 with e.g.

$$p(g) = \begin{cases} \sigma_{(1\,2)} & \text{on } A_8^{(1)} A_8^{(2)}, \\ \tau_{(3)} & \text{on } A_8^{(3)} \end{cases}$$

so that $p(g)$ has to be in the third conjugacy class.

- (4) For $L = N(E_8^3)$ the group $H \cong S_3$ has exactly one conjugacy class of order 2, which has length 3.
- (5) For $L = N(A_5^4 D_4)$ the group $H \cong \text{GL}(2, 3)$ has exactly two conjugacy classes of order 2. These can be distinguished by their projection to G_2 :

conj. class in H		proj. to G_2	
order	length	on A_5^4	on D_4
2	1	id	id
2	12	$\sigma_{(1\,2)}, \dots$	id

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong A_5^4 D_4$ of order 2 with fixed-point Lie subalgebra $A_1 A_2^2 A_3 A_5 B_2 \mathbb{C}$ can only occur as

$$A_5^2 \rightsquigarrow A_5, \quad A_5 \rightsquigarrow A_2^2 \mathbb{C}, \quad A_5 \rightsquigarrow A_3, \quad D_4 \rightsquigarrow A_1 B_2$$

up to reordering of the four copies of A_5 with e.g.

$$q(g) = \begin{cases} \sigma_{(1\,2)} & \text{on } A_5^4, \\ \text{id} & \text{on } D_4 \end{cases}$$

so that $p(g)$ has to be in the second conjugacy class.

- (6) For $L = N(E_6^4)$ the group $H \cong \text{GL}(2, 3)$ has exactly two conjugacy classes of order 2. These can be distinguished by their projection to G_2 :

conj. class in H		proj. to G_2
order	length	on E_6^4
2	1	id
2	12	$\sigma_{(1\,2)}, \dots$

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong E_6^4$ of order 2 with fixed-point Lie subalgebra $C_4^2 F_4^2$ can only occur as

$$E_6 \rightsquigarrow C_4, \quad E_6 \rightsquigarrow C_4, \quad E_6 \rightsquigarrow F_4, \quad E_6 \rightsquigarrow F_4$$

up to reordering of the four copies of E_6 with

$$q(g) = \text{id}$$

so that $p(g)$ has to be in the first conjugacy class.

- (7) For $L = N(A_4^6)$ the group H (a group extension $\mathbb{Z}_2.S_5$) has exactly three conjugacy classes of order 2. In this case it is not enough to study the projection to G_2 . The conjugacy classes can be described as:

conj. class in H		
order	length	on A_4^6
2	1	$\tau_{(1)}\tau_{(2)}\tau_{(3)}\tau_{(4)}\tau_{(5)}\tau_{(6)}$
2	15	$\sigma_{(1\ 2)}\sigma_{(3\ 4)}, \dots$
2	15	$\sigma_{(1\ 2)}\sigma_{(3\ 4)}\tau_{(5)}\tau_{(6)}, \dots$

where $\tau_{(i)}$ denotes the diagram automorphism τ on the i -th copy $A_4^{(i)}$ of A_4 , $i = 1, \dots, 6$.

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong A_4^6$ of order 2 with fixed-point Lie subalgebra $A_4^2 B_2^2$ can only occur as

$$A_4^2 \rightsquigarrow A_4, \quad A_4^2 \rightsquigarrow A_4, \quad A_4 \rightsquigarrow B_2, \quad A_4 \rightsquigarrow B_2$$

up to reordering of the six copies of A_4 with e.g.

$$p(g) = \begin{cases} \sigma_{(1\ 2)} & \text{on } A_8^{(1)} A_8^{(2)}, \\ \sigma_{(3\ 4)} & \text{on } A_8^{(3)} A_8^{(4)}, \\ \tau_{(5)} & \text{on } A_8^{(5)}, \\ \tau_{(6)} & \text{on } A_8^{(6)} \end{cases}$$

so that $p(g)$ has to be in the second conjugacy class.

- (8) For $L = N(A_2^{12})$ the group H (a group extension $\mathbb{Z}_2.M_{12}$) has exactly three conjugacy classes of order 2. Also in this case it is not enough to study the projection to G_2 . The conjugacy classes can be described as:

conj. class in H		
order	length	on A_2^{12}
2	1	$\tau_{(1)}\tau_{(2)} \dots \tau_{(12)}$
2	495	$\sigma_{(1\ 2)}\sigma_{(3\ 4)}\sigma_{(5\ 6)}\sigma_{(7\ 8)}, \dots$
2	495	$\sigma_{(1\ 2)}\sigma_{(3\ 4)}\sigma_{(5\ 6)}\sigma_{(7\ 8)}\tau_{(9)}\tau_{(10)}\tau_{(11)}\tau_{(12)}, \dots$

where $\tau_{(i)}$ denotes the diagram automorphism τ on the i -th copy $A_2^{(i)}$ of A_2 , $i = 1, \dots, 12$.

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong A_2^{12}$ of order 2 with fixed-point Lie subalgebra $A_1^4 A_2^4$ can only occur as

$$A_2^2 \rightsquigarrow A_2 \text{ (4 times)}, \quad A_2 \rightsquigarrow A_1 \text{ (4 times)}$$

up to reordering of the twelve copies of A_2 with e.g.

$$p(g) = \begin{cases} \sigma_{(1\ 2)} & \text{on } A_8^{(1)} A_8^{(2)}, \\ \sigma_{(3\ 4)} & \text{on } A_8^{(3)} A_8^{(4)}, \\ \sigma_{(5\ 6)} & \text{on } A_8^{(5)} A_8^{(6)}, \\ \sigma_{(7\ 8)} & \text{on } A_8^{(7)} A_8^{(8)}, \\ \tau_{(9)} & \text{on } A_8^{(9)}, \\ \tau_{(10)} & \text{on } A_8^{(10)}, \\ \tau_{(11)} & \text{on } A_8^{(11)}, \\ \tau_{(12)} & \text{on } A_8^{(12)} \end{cases}$$

so that $p(g)$ has to be in the third conjugacy class.

- (9) For $L = N(A_{11} D_7 E_6)$ the group $H \cong \mathbb{Z}_2$ has exactly one element of order 2.

- (10) As in the previous case.
- (11) For $L = N(A_4^6)$ the group H (a group extension $\mathbb{Z}_2.S_5$) has exactly one conjugacy class of order 5, which has length 24.
- (12) For $L = N(D_{10}E_7^2)$ the group $H \cong \mathbb{Z}_2$ has exactly one element of order 2.
- (13) As in the previous case.
- (14) For $L = N(E_6^4)$ the group $H \cong \mathrm{GL}(2, 3)$ has exactly one conjugacy class of order 6, which has length 8.
- (15) For $L = N(D_4^6)$ the group H (a group extension $\mathbb{Z}_3.S_6$) has exactly two conjugacy classes of order 4. These can be distinguished by their projection to G_2 :

conj. class in H		proj. to G_2
order	length	on D_4^6
4	90	$\sigma_{(1\,2\,3\,4)}\sigma_{(5\,6)}, \dots$
4	270	$\sigma_{(1\,2\,3\,4)}, \dots$

By Kac's theory, the automorphism $r(g)$ of $(V_L)_1 \cong D_4^6$ of order 4 with fixed-point Lie subalgebra $A_3\mathbb{C}^7$ can only occur as

$$D_4^4 \rightsquigarrow A_3\mathbb{C}, \quad D_4 \rightsquigarrow \mathbb{C}^3, \quad D_4 \rightsquigarrow \mathbb{C}^3$$

up to reordering of the six copies of D_4 with e.g.

$$q(g) = \begin{cases} \sigma_{(1\,2\,3\,4)} & \text{on } D_4^{(1)} \dots D_4^{(4)}, \\ \mathrm{id} & \text{on } D_4^{(5)}, \\ \mathrm{id} & \text{on } D_4^{(6)} \end{cases}$$

so that $p(g)$ has to be in the second conjugacy class. \square

Instead of just studying conjugacy classes in $\mathrm{Aut}((V_L)_1)$ as in Lemma 7.6 we now study conjugacy in $\mathrm{Aut}((V_L)_1)$ under automorphisms in $\pi^{-1}(H)$, i.e. under those automorphisms of $(V_L)_1$ which lift to automorphisms of V_L (see the diagram (9)), for example under inner automorphisms.

Lemma 7.8. *Under the assumptions of Proposition 7.5 the restriction $r(g) \in \mathrm{Aut}((V_L)_1)$ lies in a unique class in $\mathrm{Aut}((V_L)_1)$ up to conjugacy under automorphisms in $\pi^{-1}(H)$.*

Proof. Let h, g be two automorphisms of V_L of order n and with weight-one fixed-points $(V_L^h)_1 \cong (V_L^g)_1$ as in the assumptions of Proposition 7.5. We want to show that $r(h)$ and $r(g)$ in $\mathrm{Aut}((V_L)_1)$ are conjugate under an automorphism in $\pi^{-1}(H)$. For this we make a case-by-case analysis using Lemmas 7.1, 7.4, 7.6 and 7.7.

First note that by Lemma 7.7 we know that $p(g)$ and $p(h)$ are conjugate in H , i.e. h is conjugate to xg in $\mathrm{Aut}(V_L)$ for some inner automorphism $x \in K = \langle \{e^{v_0} \mid v \in (V_L)_1\} \rangle$. Restricting to $(V_L)_1$ we can state that $r(h)$ is conjugate to $xr(g)$ for some $x \in \mathrm{Inn}((V_L)_1)$, conjugate under an automorphism in $r(\mathrm{Aut}(V_L)) = \pi^{-1}(H)$. Hence, we may assume in the proof that $r(h) = xr(g)$ for some $x \in \mathrm{Inn}((V_L)_1)$.

- (9) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_{11}D_7E_6$ of order 2 with fixed-point Lie subalgebra $A_1B_5D_6F_4$ can only occur as

$$A_{11} \rightsquigarrow D_6, \quad D_7 \rightsquigarrow A_1B_5, \quad E_6 \rightsquigarrow C_4.$$

We can study the restriction to each simple component separately. In all three cases there is a unique conjugacy class of automorphisms of order 2 of the simple component with the given fixed-point Lie subalgebra. Since all of the automorphisms must be outer, by Lemma 7.4 there is even a unique class up to conjugacy under inner automorphisms.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of $A_{11}D_7E_6$.

- (10) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_{11}D_7E_6$ of order 2 with fixed-point Lie subalgebra $B_2B_4C_4C_6$ can only occur as

$$A_{11} \rightsquigarrow C_6, \quad D_7 \rightsquigarrow B_2B_4, \quad E_6 \rightsquigarrow C_4.$$

With exactly the same reasoning as in case (9) we conclude that $r(h)$ and $r(g)$ are conjugate under an inner automorphism of $A_{11}D_7E_6$.

- (6) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong E_6^4$ of order 2 with fixed-point Lie subalgebra $C_4^2F_4^2$ can only occur as

$$E_6 \rightsquigarrow C_4, \quad E_6 \rightsquigarrow C_4, \quad E_6 \rightsquigarrow F_4, \quad E_6 \rightsquigarrow F_4$$

up to reordering of the four copies of E_6 .

Since the group $G_2 \cong S_4$ is the full symmetric group permuting the four copies of E_6 , it is possible to find elements $y_1, y_2 \in H$ such that conjugating $r(h)$ and $r(g)$ by elements in $\pi^{-1}(\{y_1\})$ and $\pi^{-1}(\{y_2\})$, respectively, puts $r(h)$ and $r(g)$ into such a form that their fixed points coincide and occur exactly in the above specified order.

Now we can study the restriction of $r(h)$ and $r(g)$ to each simple component separately. With exactly the same reasoning as in case (9) we conclude that $r(h)$ and $r(g)$ are conjugate under an automorphism of E_6^4 in $\pi^{-1}(H)$.

- (3) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_8^3$ of order 2 with fixed-point Lie subalgebra A_8B_4 can only occur as

$$A_8^2 \rightsquigarrow A_8, \quad A_8 \rightsquigarrow B_4$$

up to reordering of the three copies of A_8 .

Recall that we may assume that $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$. But then the fixed points of $r(h)$ and $r(g)$ must occur in the same order, and without loss of generality we may assume that they occur in the order specified above.

Then by Lemma 7.1 the restrictions of $r(h)$ and $r(g)$ to the first two copies of A_8 are conjugate under an inner automorphism. On the third copy of A_8 the automorphisms $r(g)$ and $r(h)$ are outer and hence conjugate under an inner automorphism by Lemma 7.4.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of A_8^3 .

- (4) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong E_8^3$ of order 2 with fixed-point Lie subalgebra D_8E_8 can only occur as

$$E_8^2 \rightsquigarrow E_8, \quad E_8 \rightsquigarrow D_8$$

up to reordering of the three copies of E_8 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above.

Then by Lemma 7.1 the restrictions of $r(h)$ and $r(g)$ to the first two copies of E_8 are conjugate under an inner automorphism. On the third copy of E_8 the automorphisms $r(g)$ and $r(h)$ are conjugate and hence conjugate under an inner automorphism since $\text{Out}(E_8)$ is trivial.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of E_8^3 .

- (7) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_4^6$ of order 2 with fixed-point Lie subalgebra $A_4^2B_2^2$ can only occur as

$$A_4^2 \rightsquigarrow A_4, \quad A_4^2 \rightsquigarrow A_4, \quad A_4 \rightsquigarrow B_2, \quad A_4 \rightsquigarrow B_2$$

up to reordering of the six copies of A_4 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above.

As before, by Lemma 7.1, the restrictions of $r(h)$ and $r(g)$ to the first two and second two copies of A_4 lie in the same class of automorphisms up to conjugation by inner automorphisms.

Since the restrictions of $r(h)$ and $r(g)$ to the second last and last copy of A_4 must be outer, by Lemma 7.4 they are conjugate under an inner automorphism.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of A_4^6 .

- (8) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_2^{12}$ of order 2 with fixed-point Lie subalgebra $A_1^4 A_2^4$ can only occur as

$$A_2^2 \rightsquigarrow A_2 \text{ (4 times)}, \quad A_2 \rightsquigarrow A_1 \text{ (4 times)}$$

up to reordering of the twelve copies of A_2 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above.

Exactly as in the previous case, using Lemmas 7.1 and 7.4, we conclude that $r(h)$ and $r(g)$ are conjugate under an inner automorphism of A_2^{12} .

- (11) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_4^6$ of order 5 with fixed-point Lie subalgebra $A_4 \mathbb{C}^4$ can only occur as

$$A_4^5 \rightsquigarrow A_4, \quad A_4 \rightsquigarrow \mathbb{C}^4$$

up to reordering of the six copies of A_4 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above.

As before, by Lemma 7.1, the restrictions of $r(h)$ and $r(g)$ to the first five copies of A_4 lie in the same class of automorphisms up to conjugation by inner automorphisms.

An automorphism of A_4 with fixed-point Lie subalgebra \mathbb{C}^4 is necessarily inner. All automorphisms of order 5 of A_4 with fixed-point Lie subalgebra \mathbb{C}^4 are conjugate and even conjugate under an inner automorphism (see Section 1.7).

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of A_4^6 .

- (2) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_7^2 D_5^2$ of order 2 with fixed-point Lie subalgebra $A_1^2 A_3 A_7 B_2^2$ can only occur as

$$A_7^2 \rightsquigarrow A_7, \quad D_5 \rightsquigarrow A_1^2 A_3, \quad D_5 \rightsquigarrow B_2^2$$

up to reordering of the two copies of D_5 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above. Note that on the first copy of D_5 the automorphisms are inner, and on the second they are outer.

The restrictions of $r(h)$ and $r(g)$ to A_7^2 lie in the same class of automorphisms up to conjugation by inner automorphisms by Lemma 7.1.

The restrictions of $r(h)$ and $r(g)$ to the second copy of D_5 are outer and hence lie in the same class up to conjugacy under inner automorphisms by Lemma 7.4.

Restricted to the first copy of D_5 the automorphisms $r(h)$ and $r(g)$ are inner, but all automorphisms of order 2 of D_5 with fixed-point Lie subalgebra $A_1^2 A_3$ are conjugate and even conjugate under an inner automorphism (see Section 1.7).

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of $A_7^2 D_5^2$.

- (5) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_5^4 D_4$ of order 2 with fixed-point Lie subalgebra $A_1 A_2^2 A_3 A_5 B_2 \mathbb{C}$ can only occur as

$$A_5^2 \rightsquigarrow A_5, \quad A_5 \rightsquigarrow A_2^2 \mathbb{C}, \quad A_5 \rightsquigarrow A_3, \quad D_4 \rightsquigarrow A_1 B_2$$

up to reordering of the four copies of A_5 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above. Note that on the third copy of A_5 the automorphisms are inner, and on the fourth they are outer.

The restrictions of $r(h)$ and $r(g)$ to A_5^2 lie in the same class of automorphisms up to conjugation by inner automorphisms by Lemma 7.1.

Restricted to the third copy of A_5 the automorphisms $r(h)$ and $r(g)$ are inner, but all automorphisms of order 2 of A_5 with fixed-point Lie subalgebra $A_2^2 \mathbb{C}$ are conjugate and even conjugate under an inner automorphism (see Section 1.7).

The restrictions of $r(h)$ and $r(g)$ to the fourth copy of A_5 are outer and hence lie in the same class up to conjugacy under inner automorphisms by Lemma 7.4.

The restrictions of $r(h)$ and $r(g)$ to D_4 are outer and square to inner automorphisms. By Kac's theory $r(h)$ and $r(g)$ are conjugate on D_4 . Since we may assume that $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, both $r(h)$ and $r(g)$ project to the same transposition in $\text{Out}(D_4) \cong S_3$. This implies that they can only be conjugate under an element that projects to the identity or to this transposition. In fact, they are conjugate under both.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of $A_5^4 D_4$.

- (1) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong A_9^2 D_6$ of order 2 with fixed-point Lie subalgebra $A_5 C_5 D_5 \mathbb{C}$ can only occur as

$$A_9 \rightsquigarrow C_5, \quad A_9 \rightsquigarrow D_5, \quad D_6 \rightsquigarrow A_5 \mathbb{C}$$

up to reordering of the two copies of A_9 .

Since the group $G_2 \cong S_2$ is the full symmetric group permuting the two copies of A_9 , it is possible to find elements $y_1, y_2 \in H$ such that conjugating $r(h)$ and $r(g)$ by elements in $\pi^{-1}(\{y_1\})$ and $\pi^{-1}(\{y_2\})$, respectively, puts $r(h)$ and $r(g)$ into such a form that their fixed points coincide and occur exactly in the above specified order.

For each of the three simple components there is a unique conjugacy class of its automorphisms with the given fixed-point Lie subalgebra. On the two copies of A_9 the automorphisms must be outer, and by Lemma 7.4 there is a unique class up to conjugacy under inner automorphisms.

The automorphism on D_6 is inner. While there is a unique conjugacy class of automorphisms of D_6 of order 2 with fixed-point Lie subalgebra $A_5 \mathbb{C}$, two such automorphisms need not be conjugate under an inner automorphism. Indeed, the conjugacy class in $\text{Aut}(D_6)$ splits up into two classes with respect to conjugacy under $\text{Inn}(D_6)$. If the restrictions of $r(h)$ and $r(g)$ to D_6 are conjugate under an inner automorphism, we are done. So let us assume they are conjugate under an outer automorphism, i.e. under

an automorphism whose projection to $\text{Out}(D_6)$ is τ . The group $H \cong \mathbb{Z}_4$ contains two elements of order 4, namely $\sigma_{(1\ 2)}\tau_{(1)}\tau_{(3)}$ and $\sigma_{(1\ 2)}\tau_{(2)}\tau_{(3)}$. Let $y \in H$ be one of those two elements. Conjugating $r(h)$ by an element in $\pi^{-1}(\{y\})$ does not change the order of its fixed points on the two copies of A_9 and it brings it to a form such that the restrictions of $r(h)$ and $r(g)$ to D_6 are conjugate under an inner automorphism. So again we are done.

Hence $r(h)$ and $r(g)$ are conjugate under an automorphism of $A_9^2 D_6$ in $\pi^{-1}(H)$.

- (12) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong D_{10}E_7^2$ of order 4 with fixed-point Lie subalgebra $A_1A_4A_7B_3\mathbb{C}$ can only occur as

$$D_{10} \rightsquigarrow A_1A_4B_3\mathbb{C}, \quad E_7^2 \rightsquigarrow A_7.$$

The restrictions of $r(h)$ and $r(g)$ to E_7^2 are conjugate under some element in $\text{Aut}(E_7^2)$. First assume that this element is inner. Then, since the restrictions of $r(h)$ and $r(g)$ to D_{10} are outer and hence conjugate under an inner automorphism by Lemma 7.4, $r(h)$ and $r(g)$ are in total conjugate under an inner automorphism.

If on the other hand the restrictions of $r(h)$ and $r(g)$ to E_7^2 are conjugate under an element of $\text{Aut}(E_7^2)$ that projects to $\sigma_{(2\ 3)} \in \text{Out}(E_7^2)$, then we note that since the restrictions of $r(h)$ and $r(g)$ to D_{10} are outer, they are by Lemma 7.4 also conjugate under an outer automorphism. So $r(h)$ and $r(g)$ are in total conjugate under an element in $\text{Aut}(D_{10}E_7^2)$ that projects to $\tau_{(1)}\sigma_{(2\ 3)} \in \text{Out}(D_{10}E_7^2)$. Since $H = \langle \tau_{(1)}\sigma_{(2\ 3)} \rangle \cong \mathbb{Z}_2$, this implies that $r(h)$ and $r(g)$ are conjugate under an automorphism of $D_{10}E_7^2$ in $\pi^{-1}(H)$.

- (13) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong D_{10}E_7^2$ of order 4 with fixed-point Lie subalgebra $A_7B_2B_6\mathbb{C}$ can only occur as

$$D_{10} \rightsquigarrow B_2B_6\mathbb{C}, \quad E_7^2 \rightsquigarrow A_7.$$

With exactly the same reasoning as in case (12) we conclude that $r(h)$ and $r(g)$ are conjugate under an automorphism of $D_{10}E_7^2$ in $\pi^{-1}(H)$.

- (14) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong E_6^4$ of order 6 with fixed-point Lie subalgebra $A_1A_2C_4\mathbb{C}$ can only occur as

$$E_6^3 \rightsquigarrow C_4, \quad E_6 \rightsquigarrow A_1A_2\mathbb{C}$$

up to reordering of the four copies of E_6 .

Once again we may take $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, and so without loss of generality the fixed points of $r(g)$ and $r(h)$ occur in the order specified above.

The restrictions of $r(h)$ and $r(g)$ to the fourth copy of E_6 are outer and hence lie in the same class up to conjugacy under inner automorphisms by Lemma 7.4.

We slightly modify the argument of Lemma 7.1 in order to show that the restrictions of $r(h)$ and $r(g)$ to the first three copies of E_6 are conjugate under an inner automorphism. We can write $r(g)$ without loss of generality as

$$(v_1, v_2, v_3) \mapsto (\varphi_3 v_3, \varphi_1 v_1, \varphi_2 v_2)$$

for $(v_1, v_2, v_3) \in E_6^3$ with $\varphi_1, \varphi_2, \varphi_3 \in \text{Aut}(E_6)$ and $\varphi_3\varphi_2\varphi_1$ outer, of order 2 and with fixed points C_4 . Then, writing $x = (x_1, x_2, x_3)$ with $x_1, x_2, x_3 \in \text{Inn}(E_6)$, $r(h) = xr(g)$ is given by

$$(v_1, v_2, v_3) \mapsto (x_1\varphi_3 v_3, x_2\varphi_1 v_1, x_3\varphi_2 v_2)$$

for $(v_1, v_2, v_3) \in E_6^3$. Also $x_1\varphi_3x_3\varphi_2x_2\varphi_1$ is outer, of order 2 and has fixed points C_4 . By Kac's theory $\varphi_3\varphi_2\varphi_1$ and $x_1\varphi_3x_3\varphi_2x_2\varphi_1$ are conjugate in

$\text{Aut}(E_6)$, and by Lemma 7.4 they are even conjugate under an inner automorphism, which we call $\xi \in \text{Inn}(E_6)$, i.e. $\xi^{-1}x_1\varphi_3x_3\varphi_2x_2\varphi_1\xi = \varphi_3\varphi_2\varphi_1$. A simple calculation then shows that

$$y^{-1}r(h)y = r(g)$$

with $y = (\xi, x_2\varphi_1\xi\varphi_1^{-1}, x_3\varphi_2x_2\varphi_1\xi\varphi_1^{-1}\varphi_2^{-1})$. Since ξ, x_2, x_3 are inner, so is y , i.e. the projection of y to $\text{Out}(E_6^3)$ is trivial.

- Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of E_6^4 .
 (15) By Kac's theory, the automorphisms $r(h)$ and $r(g)$ of $(V_L)_1 \cong D_4^6$ of order 4 with fixed-point Lie subalgebra $A_3\mathbb{C}^7$ can only occur as

$$D_4^4 \rightsquigarrow A_3\mathbb{C}, \quad D_4 \rightsquigarrow \mathbb{C}^3, \quad D_4 \rightsquigarrow \mathbb{C}^3$$

up to reordering of the six copies of D_4 .

Since we may assume that $r(h) = xr(g)$ for some $x \in \text{Inn}((V_L)_1)$, in this case the fixed-points of $r(h)$ and $r(g)$ occur in the same order, and we may as well assume that they occur in the above specified order.

Then by Lemma 7.1 $r(h)$ is conjugate to $r(g)$ under an inner automorphism restricted to the first four copies of D_4 . On the fifth and sixth copy of D_4 the automorphisms $r(g)$ and $r(h)$ are outer and hence conjugate under an inner automorphism by Lemma 7.4.

Hence $r(h)$ and $r(g)$ are conjugate under an inner automorphism of D_4^6 . \square

Proof of Proposition 7.5. In each case let $\hat{\nu}$ be a standard lift to $\text{Aut}(V_L)$ of the lattice automorphism $\nu \in \text{Aut}(L)$ specified in Proposition 5.1. Since $\hat{\nu}$ has the required order, weight-one fixed-point Lie subalgebra and, if necessary, conformal weight of its twisted module, the existence of the automorphism is established.

We will show that any automorphism $g \in \text{Aut}(V_L)$ with these properties is conjugate to $\hat{\nu}$. We showed in the previous Lemma that $r(g)$ and $r(\hat{\nu})$ are conjugate in $\text{Aut}((V_L)_1)$ under an automorphism in $\pi^{-1}(H)$.

This automorphism in $\pi^{-1}(H)$ can be lifted to an automorphism in $\text{Aut}(V_L)$, and using Lemma 7.2 this implies that g is conjugate in $\text{Aut}(V_L)$ to $e^{(2\pi i)h_0}\hat{\nu}$ for some $h \in Q'$, the dual lattice of Q . In fact, by Lemma 7.3 we may assume that $g = e^{(2\pi i)h_0}\hat{\nu}$ for some $h \in \pi_\nu(Q')$. Note that now $e^{(2\pi i)h_0}$ and $\hat{\nu}$ commute.

We aim to show that $e^{(2\pi i)h_0} = \text{id}$. Since L is unimodular, $e^{(2\pi i)h_0} = \text{id}$ for $h \in L$ so that we only need to consider representatives h for $\pi_\nu(Q')/\pi_\nu(L)$. Moreover, we only need to consider those elements h such that $g = e^{(2\pi i)h_0}\hat{\nu}$ has the same order as $\hat{\nu}$. This is the case if and only if $\text{ord}(\hat{\nu})h \in L$.

Now we proceed case by case:

- (1) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_2 . For trivial h we obtain $\rho(V_L(g)) = 1$ while for non-trivial h we obtain $\rho(V_L(g)) = 5/4$. Hence, if we demand that $\rho(V_L(g)) = 1$, g has to be conjugate to $\hat{\nu}$.
- (2) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_4 , but only the trivial element preserves the order of $\hat{\nu}$. Hence, g is conjugate to $\hat{\nu}$.
- (3) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_3 , but only the trivial element preserves the order of $\hat{\nu}$. Hence, g is conjugate to $\hat{\nu}$.
- (4) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial since $L = Q$, and so g is conjugate to $\hat{\nu}$.
- (5) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_6 , but only the trivial element preserves the order of $\hat{\nu}$. Hence, g is conjugate to $\hat{\nu}$.
- (6) Also in this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial. Hence, g is conjugate to $\hat{\nu}$.
- (7) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_5 , but only the trivial element preserves the order of $\hat{\nu}$. Hence, g is conjugate to $\hat{\nu}$.

- (8) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, but only the trivial element preserves the order of $\hat{\nu}$. Hence, g is conjugate to $\hat{\nu}$.
- (9) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial. Hence, g is conjugate to $\hat{\nu}$.
- (10) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_2 . For trivial h we obtain $\rho(V_L(g)) = 1$ while for non-trivial h we obtain $\rho(V_L(g)) = 5/4$. Hence, if we demand that $\rho(V_L(g)) = 1$, g has to be conjugate to $\hat{\nu}$.
- (11) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_5 . For trivial h we obtain $\rho(V_L(g)) = 1$ while for the four non-trivial choices of h we obtain $\rho(V_L(g)) = 27/25$ or $\rho(V_L(g)) = 28/25$. Hence, if we demand that $\rho(V_L(g)) = 1$, g has to be conjugate to $\hat{\nu}$.
- (12a) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial. Hence, g is conjugate to $\hat{\nu}$.
- (12b) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial. Hence, g is conjugate to $\hat{\nu}$.
- (13) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial. Hence, g is conjugate to $\hat{\nu}$.
- (14) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is trivial (we see from the cycle shape of ν that π_ν is the zero map). Hence, g is conjugate to $\hat{\nu}$.
- (15) In this case $\pi_\nu(Q')/\pi_\nu(L)$ is isomorphic to \mathbb{Z}_2 . For trivial h we obtain $\rho(V_L(g)) = 1$ while for the non-trivial choice of h we obtain $\rho(V_L(g)) = 17/16$. Hence, if we demand that $\rho(V_L(g)) = 1$, g has to be conjugate to $\hat{\nu}$.

In order to prove the second statement of the proposition we note that the inverse orbifold automorphism $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_h)})$ with respect to the inner automorphism σ_h (see table on p. 19) has the same order and weight-one fixed-points as $\hat{\nu}$. In all cases except (1), (10), (11) and (15) this shows that $\hat{\nu} \in \text{Aut}(V_L)$ and $\zeta \in \text{Aut}(V^{\text{orb}(\sigma_h)})$ are conjugate under the isomorphism $V^{\text{orb}(\sigma_h)} \cong V_L$.

For the four remaining cases we still need to show that $\rho(V^{\text{orb}(\sigma_h)}(\zeta)) = 1$. The ζ -twisted $V^{\text{orb}(\sigma_h)}$ -module $V^{\text{orb}(\sigma_h)}(\zeta)$ decomposes as

$$V^{\text{orb}(\sigma_h)}(\zeta) = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,1)}$$

as described in Section 2.1. We know that $\rho(V(\sigma_h^i)) \geq 1$ for all $i \in \mathbb{Z}_n \setminus \{0\}$, and hence, since $V(\sigma_h^i) = \bigoplus_{j \in \mathbb{Z}_n} W^{(i,j)}$, $\rho(W^{(i,j)}) \geq 1$ for $i \in \mathbb{Z}_n \setminus \{0\}$ and $j \in \mathbb{Z}_n$. Moreover, since $V = \bigoplus_{j \in \mathbb{Z}_n} W^{(0,j)}$ is of CFT-type, $\rho(W^{(0,j)}) \geq 1$ for $j \in \mathbb{Z}_n \setminus \{0\}$. Hence, it suffices to show that $\rho(W^{(0,1)}) = 1$ or in other words that $\{v \in V_1 \mid \sigma_h v = \xi_n v\}$ is non-empty. This is easily verified for the cases at hand. \square

8. MAIN RESULT

Combining the results of Propositions 5.1, 6.6 and 7.5 and following the procedure described in Section 4.2 we arrive at the main result of this text:

Theorem 8.1 (Uniqueness). *A tame, holomorphic vertex operator algebra of central charge 24 is up to isomorphism uniquely determined by the Lie algebra structure of its weight-one space V_1 if V_1 is isomorphic to $A_5 C_5 E_{6,2}$, $A_3 A_{7,2} C_3^2$, $A_{8,2} F_{4,2}$, $B_8 E_{8,2}$, $A_{2,2}^2 A_{5,2} B_2$, $C_8 F_4^2$, $A_{4,2}^2 C_{4,2}$, $A_{2,2}^4 D_{4,4}$, $B_5 E_{7,2} F_4$, $B_4 C_6^2$, $A_{4,5}^2$, $A_4 A_{9,2} B_3$, $B_6 C_{10}$, $A_1 C_{5,3} G_{2,2}$ or $A_{1,2} A_{3,4}^3$, corresponding to the cases 44, 33, 36, 62, 26, 52, 22, 13, 53, 48, 9, 40, 56, 21 and 7 on Schellekens' list.*

The uniqueness of the case $B_8 E_{8,2}$ (case 62) is already proved in [LS15] using framed vertex operator algebras. Moreover, Lam and Shimakura have announced to have proved the uniqueness of $A_{4,5}^2$ (case 9) and $A_{1,2} A_{3,4}^3$ (case 7) starting from the Leech lattice vertex operator algebra.

REFERENCES

- [Bor86] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986. <http://math.berkeley.edu/~reb/papers/va/va.pdf>.
- [CM16] Scott Carnahan and Masahiko Miyamoto. Regularity of fixed-point vertex operator subalgebras. (arXiv:1603.05645v3 [math.RT]), 2016.
- [CS99] John H. Conway and Neil J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren Math. Wiss.* Springer, 3rd edition, 1999.
- [DGM90] Louise Dolan, Peter Goddard and Paul S. Montague. Conformal field theory, triality and the Monster group. *Phys. Lett. B*, 236(2):165–172, 1990.
- [DGM96] Louise Dolan, Peter Goddard and Paul S. Montague. Conformal field theories, representations and lattice constructions. *Comm. Math. Phys.*, 179(1):61–120, 1996. (arXiv:hep-th/9410029v1).
- [DLM00] Chongying Dong, Haisheng Li and Geoffrey Mason. Modular-invariance of trace functions in orbifold theory and generalized moonshine. *Commun. Math. Phys.*, 214:1–56, 2000. (arXiv:q-alg/9703016v2).
- [DM97] Chongying Dong and Geoffrey Mason. On quantum Galois theory. *Duke Math. J.*, 86(2):305–321, 1997. (arXiv:hep-th/9412037v1).
- [DM04a] Chongying Dong and Geoffrey Mason. Holomorphic vertex operator algebras of small central charge. *Pacific J. Math.*, 213(2):253–266, 2004. (arXiv:math/0203005v1 [math.QA]).
- [DM04b] Chongying Dong and Geoffrey Mason. Rational vertex operator algebras and the effective central charge. *Int. Math. Res. Not.*, 2004(56):2989–3008, 2004. (arXiv:math/0201318v1 [math.QA]).
- [DM06] Chongying Dong and Geoffrey Mason. Integrability of C_2 -cofinite vertex operator algebras. *Int. Math. Res. Not.*, pages 1–15, Art. ID 80468, 2006. (arXiv:math/0601569v1 [math.QA]).
- [DN99] Chongying Dong and Kiyokazu Nagatomo. Automorphism groups and twisted modules for lattice vertex operator algebras. In Naihuan Jing and Kailash C. Misra, editors, *Recent developments in quantum affine algebras and related topics*, volume 248 of *Contemp. Math.*, pages 117–133. Amer. Math. Soc., 1999. (arXiv:math/9808088v1 [math.QA]).
- [Don93] Chongying Dong. Vertex algebras associated with even lattices. *J. Alg.*, 161(1):245–265, 1993.
- [EMS15] Jethro van Ekeren, Sven Möller and Nils R. Scheithauer. Construction and classification of holomorphic vertex operator algebras. (arXiv:1507.08142v2 [math.RT]), 2015.
- [FLM88] Igor B. Frenkel, James I. Lepowsky and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure Appl. Math.* Academic Press, 1988.
- [FZ92] Igor B. Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66(1):123–168, 1992.
- [Hum73] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Grad. Texts in Math.* Springer, 1973. Second printing, revised.
- [ISS15] Motohiro Ishii, Daisuke Sagaki and Hiroki Shimakura. Automorphisms of Niemeier lattices for Miyamoto’s \mathbb{Z}_3 -orbifold construction. *Math. Z.*, 280(1-2):55–83, 2015. (arXiv:1402.0143v2 [math.QA]).
- [Kac90] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, 3rd edition, 1990.
- [KLL16] Kazuya Kawasetsu, Ching Hung Lam and Xingjun Lin. \mathbb{Z}_2 -orbifold construction associated with (-1) -isometry and uniqueness of holomorphic vertex operator algebras of central charge 24. (arXiv:1611.07655v1 [math.QA]), 2016.
- [Lam11] Ching Hung Lam. On the constructions of holomorphic vertex operator algebras of central charge 24. *Comm. Math. Phys.*, 305(1):153–198, 2011.
- [Li94] Haisheng Li. Symmetric invariant bilinear forms on vertex operator algebras. *J. Pure Appl. Algebra*, 96(3):279–297, 1994.
- [Li96] Haisheng Li. Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules. In Chongying Dong and Geoffrey Mason, editors, *Moonshine, the Monster, and Related Topics*, volume 193 of *Contemp. Math.*, pages 203–236. Amer. Math. Soc., 1996. (arXiv:q-alg/9504022v1).
- [LL04] James I. Lepowsky and Haisheng Li. *Introduction to Vertex Operator Algebras and Their Representations*, volume 227 of *Progr. Math.* Birkhäuser, 2004.
- [LL16] Ching Hung Lam and Xingjun Lin. A holomorphic vertex operator algebra of central charge 24 with weight one Lie algebra $F_{4,6}A_{2,2}$. (arXiv:1612.08123v1 [math.QA]), 2016.

- [LS12] Ching Hung Lam and Hiroki Shimakura. Quadratic spaces and holomorphic framed vertex operator algebras of central charge 24. *Proc. Lond. Math. Soc. (3)*, 104(3):540–576, 2012. (arXiv:1010.5303v2 [math.QA]).
- [LS15] Ching Hung Lam and Hiroki Shimakura. Classification of holomorphic framed vertex operator algebras of central charge 24. *Amer. J. Math.*, 137(1):111–137, 2015. (arXiv:1209.4677v1 [math.QA]).
- [LS16a] Ching Hung Lam and Hiroki Shimakura. A holomorphic vertex operator algebra of central charge 24 whose weight one Lie algebra has type $A_{6,7}$. *Lett. Math. Phys.*, 106(11):1575–1585, 2016. (arXiv:1606.04688v2 [math.QA]).
- [LS16b] Ching Hung Lam and Hiroki Shimakura. Orbifold construction of holomorphic vertex operator algebras associated to inner automorphisms. *Comm. Math. Phys.*, 342(3):803–841, 2016. (arXiv:1501.05094v2 [math.QA]).
- [LS16c] Ching Hung Lam and Hiroki Shimakura. Reverse orbifold construction and uniqueness of holomorphic vertex operator algebras. (arXiv:1606.08979v2 [math.QA]), 2016.
- [Mas14] Geoffrey Mason. Lattice subalgebras of strongly regular vertex operator algebras. In Winfried Kohnen and Rainer Weissauer, editors, *Conformal Field Theory, Automorphic Forms and Related Topics*, pages 31–53. Springer, 2014. (arXiv:1110.0544v1 [math.QA]).
- [Miy13] Masahiko Miyamoto. A \mathbb{Z}_3 -orbifold theory of lattice vertex operator algebra and \mathbb{Z}_3 -orbifold constructions. In Kenji Iohara, Sophie Morier-Genoud and Bertrand Rémy, editors, *Symmetries, Integrable Systems and Representations*, volume 40 of *Springer Proc. Math. Stat.*, pages 319–344. Springer, 2013. (part of arXiv:1003.0237v2 [math.QA]).
- [Miy15] Masahiko Miyamoto. C_2 -cofiniteness of cyclic-orbifold models. *Comm. Math. Phys.*, 335(3):1279–1286, 2015. (arXiv:1306.5031v1 [math.QA]).
- [Mon94] Paul S. Montague. Orbifold constructions and the classification of self-dual $c = 24$ conformal field theories. *Nucl. Phys. B*, 428(1–2):233–258, 1994. (arXiv:hep-th/9403088v1).
- [MT04] Masahiko Miyamoto and Kenichiro Tanabe. Uniform product of $A_{g,n}(V)$ for an orbifold model V and G -twisted Zhu algebra. *J. Algebra*, 274(1):80–96, 2004. (arXiv:math/0112054v3 [math.QA]).
- [Mö16] Sven Möller. *A Cyclic Orbifold Theory for Holomorphic Vertex Operator Algebras and Applications*. Ph.D. thesis, Technische Universität Darmstadt, 2016. (arXiv:1611.09843v1 [math.QA]).
- [Sch93] A. N. Schellekens. Meromorphic $c = 24$ conformal field theories. *Comm. Math. Phys.*, 153(1):159–185, 1993. (arXiv:hep-th/9205072v1).
- [Sch09] Nils R. Scheithauer. The Weil representation of $SL_2(\mathbb{Z})$ and some applications. *Int. Math. Res. Not.*, 2009(8):1488–1545, 2009. <http://www.mathematik.tu-darmstadt.de/~scheithauer/papers/weil.pdf>.
- [Sch15] Nils R. Scheithauer. Some constructions of modular forms for the Weil representation of $SL_2(\mathbb{Z})$. *Nagoya Math. J.*, 220:1–43, 2015. <http://www.mathematik.tu-darmstadt.de/~scheithauer/papers/modularforms4.pdf>.
- [SS16] Daisuke Sagaki and Hiroki Shimakura. Application of a \mathbb{Z}_3 -orbifold construction to the lattice vertex operator algebras associated to Niemeier lattices. *Trans. Amer. Math. Soc.*, 368(3):1621–1646, 2016. (arXiv:1302.4826v4 [math.QA]).
- [Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.